

# ON TENSORIAL ESTIMATES, SHARP TRACE THEOREMS, AND PARALLEL SCALAR REDUCTIONS

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**ABSTRACT.** In this paper, we consider various bilinear product and integrated product estimates on a one-parameter foliation of surfaces with evolving geometries. Moreover, we wish to do this with only very weak control on how the geometries evolve. Several of these estimates were proved in [13] in very special settings. A primary objective of this paper is to significantly simplify the proofs of the main estimates found in [13]. Another goal is to generalize the estimates of [13] to more abstract settings.

## 1. INTRODUCTION

In this paper, we investigate a number of bilinear product and integrated product estimates in general  $(1 + 2)$ -dimensional geometric settings. For example, in the Euclidean setting  $I \times \mathbb{R}^2$ , where  $I = (0, 1)$ , a known integrated product estimate is

$$(1.1) \quad \left\| \int_0^t \partial_t \phi(s, \cdot) \psi(s, \cdot) ds \right\|_{B_{2,1}^0(\mathbb{R}^2)} \lesssim \|\phi\|_{H_{t,x}^1(I \times \mathbb{R}^2)} \|\psi\|_{H_{t,x}^1(I \times \mathbb{R}^2)}, \quad t \in I,$$

where  $\phi, \psi$  are sufficiently nice functions on  $I \times \mathbb{R}^2$ . This estimate was proved in [13, Sect. 3] using classical Littlewood-Paley decompositions. A corresponding non-integrated product estimate, again on  $I \times \mathbb{R}^2$ , is

$$(1.2) \quad \|\phi(t, \cdot) \psi(t, \cdot)\|_{B_{2,1}^0(\mathbb{R}^2)} \lesssim \|\phi\|_{H_{t,x}^1(I \times \mathbb{R}^2)} \|\psi\|_{H_{t,x}^1(I \times \mathbb{R}^2)}, \quad t \in I.$$

Again, this can be proved using frequency decomposition methods.

The general questions we wish to pose are the following:

- (1) Do analogues of (1.1) and (1.2), as well as other related estimates, hold when  $\mathbb{R}^2$  is replaced by another surface  $\mathcal{S}$ , and when the geometry of  $\mathcal{S}$  is allowed to change with the time variable  $t$ ?
- (2) Can we establish these estimates with only “very weak” assumptions on how the geometries of  $\mathcal{S}$  evolve with time?

In this paper, we will examine a rather general case, in which  $\mathcal{S}$  is compact. The main results of this paper are stated in Theorems 5.7-5.9. For more purely geometric versions of these bilinear product estimates, one can combine the above theorems with the comparison estimates of Theorem 6.7.

**1.1. Motivations.** The main motivation for this problem comes from mathematical general relativity, in the analysis of regular null cones in  $(1 + 3)$ -dimensional spacetimes.<sup>1</sup> More specifically, one wishes to control the geometry of the null cone given the assumption of bounded or small curvature flux. Here, “controlling the

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<sup>1</sup>Most of the existing work has been for vacuum spacetimes, though [2, 18] are exceptions.

geometry” refers to quantitative control for various connection coefficients of the cone, as well as a possible lower bound on the null conjugacy radius of the cone.

The *curvature flux* refers to an  $L^2$ -norm along the entire null cone of certain components of the spacetime curvature tensor. It is a fundamental quantity for dealing with local energy estimates involving the curvature. In particular, the curvature flux arises naturally from the Bel-Robinson tensor, which one can think of as an analogue of the stress-energy tensor for Weyl tensors.

A number of results involving mathematical relativity and the Einstein equations have relied heavily on variations of the curvature flux. Examples include the stability of Minkowski spacetimes (e.g., [2, 6, 10]), improved breakdown and continuation criteria for the Einstein equations (e.g., [16, 17, 18, 20, 24]), and the formation of black holes and trapped surfaces (e.g., [5, 15]), among several others. In particular, regarding the breakdown criteria for the Einstein equations, a major component of the proofs of this family of results was precisely that of controlling the geometry of null cones by the curvature flux.<sup>2</sup> This was by far the most technically demanding portion of the argument, involving an elaborate bootstrap procedure and the construction of a geometric tensorial Littlewood-Paley theory, cf. [12].

The required control for the null cones was first proved in [11], for geodesically foliated truncated null cones beginning from a 2-sphere in an Einstein-vacuum spacetime; technical portions of the argument were also included in [12, 13, 14]. The result was extended to null cones beginning from a point in [22, 23], which was the setting required in the proofs of the breakdown criteria. Other versions of this result include [17, 18, 24], which dealt with time foliated null cones. In particular, [18] extended the result to Einstein-scalar and Einstein-Maxwell spacetimes.

There are many reasons why this problem in general is technically demanding. The first is that the assumptions regarding the curvature flux, an  $L^2$ -norm of the curvature *along the entire null cone*, grants only very weak regularity for the geometries of the spheres foliating the cone. Because of this, many coordinate-based methods that are standard in geometric arguments fail here.<sup>3</sup> Furthermore, one has very little control on the curvatures of these spheres, which makes establishing the necessary elliptic estimates a nontrivial subproblem.

Another related difficulty is that one requires several bilinear product estimates in order to prove the required bounds, namely, the types of estimates we will discuss in this paper. In the relatively trivial Euclidean case, many of these estimates, in particular (1.1), were proved in [13, Sect. 3]. Next, the main results of [13] extended these bounds to the specific setting of geodesically foliated truncated null cones beginning from a 2-sphere in an Einstein-vacuum spacetime. In [22], similar estimates were proved for corresponding null cones beginning from a point.

These estimates in [13, 22] form the starting point for the discussions in this paper. In relation to [13, 22], our paper will accomplish two primary goals:

- We extend the results in [13, 22] to more abstract settings.
- We give proofs for our main estimates that are significantly shorter and simpler than the corresponding proofs in [13, 22].

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<sup>2</sup>More accurately, the geometry of the null cone is controlled by the curvature flux, properties of the time foliation, and properties of the matter field (in non-vacuum situations).

<sup>3</sup>In particular, the Christoffel symbols associated with coordinate systems which are transported along the null generators of the cones lack sufficient regularity.

Most of the main estimates we establish here are direct analogues of product estimates in [13, 22], though in a more general setting. In simplifying the proof, we adopt a different method for establishing the desired bounds.

In terms of applications, the immediate goal is to apply the results of this paper to other versions of the “null cone with finite curvature flux” problem. For example, one variation of interest is that of outgoing null cones extending “toward infinity”, given sufficient integral decay of the curvature flux. Using the same ideas as in [11], one hopes to obtain corresponding control for the null cone.<sup>4</sup> In particular, one could try to make sense of the Bondi mass under only the very weak assumption of curvature flux control. Another problem of potential interest is to examine time-foliated or double-null-foliated cones under similar curvature flux bounds.<sup>5</sup>

All such “variant problems”, like those mentioned above, will require corresponding bilinear product-type estimates, for precisely the same technical reasons as in [11, 17, 18, 22]. Thus, an important motivation behind the work in this paper is to establish these estimates in a sufficiently generalized setting so that they can be applied with relatively little effort to all these variant problems.

The framework developed here could also be adapted to other problems involving evolving geometries with low regularity. Indeed, many of the ideas within this paper are still applicable when the technical specifics of the problem are altered.

**1.2. The Abstract Formalism.** As noted before, the main estimates in [13] applied to an extremely specific setting: geodesically foliated null cones in vacuum spacetimes beginning from a sphere. The corresponding estimates in [22] are equally specific: same null cones, but beginning from a point. Suppose one wishes to work instead with double-null-foliated null cones or in non-vacuum backgrounds, for example. One certainly expects that the general proof template from [13] can be adapted to these cases. However, since the arguments presented in [13] (and also [22]) are highly dependent on the specific setting, then all the bilinear product estimates would in principle have to be redone. This becomes rather unsavory, since the arguments presented in [13] were quite lengthy and technically elaborate.

As a result, we wish to frame the hypotheses and conclusions of these estimates in a more abstract framework. In particular, we wish to express our results in a sufficiently general manner so that the null cone problems in [11, 13, 22], as well as potential variations of these problems, can be expressed in terms of this abstract formalism. Indeed, the settings of [11, 13, 22] can be shown to satisfy the abstract assumptions we impose for our main estimates. By similar reasoning, these assumptions are also applicable to settings found in [17, 18].

Throughout the paper, we will work with an abstract one-parameter foliation

$$\mathcal{N} = [0, \delta) \times \mathcal{S},$$

where  $\delta > 0$  is fixed, and where  $\mathcal{S}$  is a compact two-dimensional manifold. To describe the geometric information of this system, we impose a family of evolving Riemannian metrics on  $\mathcal{S}$ , parametrized by the interval  $[0, \delta)$ . In other words, for each  $0 \leq v < \delta$ , we let  $\gamma[v]$  denote a Riemannian metric on the cross-section

$$\mathcal{S}_v = \{v\} \times \mathcal{S} \simeq \mathcal{S},$$

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<sup>4</sup>In particular, the goal is to obtain corresponding decay for the Ricci coefficients.

<sup>5</sup>Time-foliated cones were in fact considered in [17, 18].

such that these  $\gamma[v]$ 's vary smoothly with respect to  $v$ . From this, we can construct other geometric objects on  $\mathcal{N}$  (e.g., covariant derivatives with respect to the  $\gamma[v]$ 's), which amount to smoothly varying aggregations of objects on the various  $\mathcal{S}_v$ 's.

The quantities we will analyze throughout the paper are represented as *horizontal tensor fields* on  $\mathcal{N}$ . These can be interpreted as smooth tensor fields on  $\mathcal{N}$  which are everywhere tangent to the  $\mathcal{S}_v$ 's. Equivalently, these can also be considered as smoothly varying one-parameter families of objects on  $\mathcal{S}$ , parametrized with respect to  $[0, \delta)$ . For instance, by aggregating the metrics  $\gamma[v]$  mentioned above into a single object  $\gamma$  on  $\mathcal{N}$ , we obtain such a horizontal tensor field. By measuring how these  $\gamma[v]$ 's evolve with respect to  $v$ , we obtain another horizontal field, which we refer to in this paper as the “second fundamental form”.

This notion of horizontal tensor fields has been used implicitly in various works in mathematical general relativity. For example, in [5, 6, 10, 11], among several others, the authors deal with various tensorial quantities on null cones that are “horizontal”, i.e., tangent to the spheres which foliate the cone. Furthermore, in [6, 16], for instance, one foliates a spacetime into Riemannian “timeslices” and analyzes tensorial objects on this spacetime which are tangent to these timeslices.

The formalisms we adopt for notating, describing, and analyzing these horizontal fields derive from [18, 19, 20], which applied these notions for similar purposes as the works mentioned above. In particular, we define our objects in terms of certain “horizontal” vector bundles on  $\mathcal{N}$  that are naturally constructed. The horizontal tensor fields can then be formulated as sections of these bundles.

One major component of the formalism is our definition of a *covariant vertical derivative* (i.e., as the parameter  $v \in [0, \delta)$  changes) of horizontal tensor fields. In comparison to previous results on null cones, e.g., [5, 6, 10, 11], one can show this coincides with the derivative operators  $\nabla_L$  and  $\nabla_{\underline{L}}$ , i.e., the appropriate horizontal projections of the corresponding *spacetime* covariant derivatives. In this sense, our construction generalizes those used in previous works on null cones.

**Remark.** *We also note that some other unrelated topics can also be connected to the abstract concepts used here. One well-known example involves the Ricci flow, in particular with a process known as “Uhlenbeck’s trick”, used to derive a covariant evolutionary equation for the Riemann curvature. In [1, Sect. 6.3], it was shown that this trick could be naturally expressed in terms of an abstract formulation equivalent to the one developed here. For further details, see also [3, 8].*

Another component is the “inverse” to the above operation: a *covariant integral* along the vertical direction. Such operations were used implicitly in [15] in order to estimate trace norms of various quantities.<sup>6</sup> Here, we shall make much more explicit mention and use of these integral operators. These will play a major role in demonstrating the improved regularity one obtains from covariant derivatives and “parallel” frames, both in the abstract setting discussed in this paper and in our specific motivating problem of null cones with bounded curvature flux.

The main observation behind our generalization of the bilinear product estimates of [13, 22] is that the required assumptions can be stated in terms of objects defined from our abstract formalism. In particular, much of the assumptions are given by

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<sup>6</sup>This is in fact very similar to the ultimate aim of this paper. The main difference with [15] is that here, we postulate far less regularity from our geometric setting.

weak integral bounds on the “second fundamental form” mentioned above. The remaining assumptions involve relatively trivial quantitative control on the geometry of a single initial leaf of the foliation  $\mathcal{N}$ . The exact conditions required for the main estimates of this paper will be described in detail in later sections.

**Remark.** *One remarkable contrast between the arguments of this paper and those in [13, 22] is that here, we make no use nor mention of the Gauss curvatures of the leaves  $\mathcal{S}_v$ . In both [13, 22], the (rather weak) regularity possessed by the Gauss curvatures played a fundamental role in the proofs.*

Finally, we mention that the process we use to prove our main estimates can be almost directly generalized to some abstract vector bundles. This is due to the observation that our covariant methods in this paper depend only on the presence of a bundle metric and a compatible bundle connection. One application of this would be for working with corresponding null cones in Einstein-Yang-Mills spacetimes. In this setting, one has precisely this bundle-metric-connection system when describing the Lie algebra-valued Yang-Mills curvature.

**1.3. Simplification of Existing Proofs.** In [13, Sect. 3], it was shown that the Euclidean analogues of these bilinear product estimates could be established with relative ease via classical Littlewood-Paley decomposition methods. Consequently, in the much less trivial setting of null cones with bounded curvature flux, i.e., of [11, 13], a reasonable first approach to similar estimates could be the following:

- Choose appropriate local coordinate systems on the null cone. <sup>7</sup>
- Apply the Euclidean estimates to these coordinate systems.
- Extract a global estimate on the null cone from these coordinate estimates.

However, such a method fails for this setting.

The fundamental reason for this failure is the extremely weak assumption of bounded curvature flux. In particular, one has only  $L^2$ -integrability of the space-time curvature in the spherical component. A cursory examination of the geometric equations involving the null cones shows that the spatial gradient of the “second fundamental form”, as defined in the preceding discussion, is of the same level as the spacetime curvature. Therefore, one expects the same spherical  $L^2$ -integrability for this gradient as for the spacetime curvature.

Now, if we are given coordinate systems as dictated by the above outline, then one can relate the Christoffel symbols associated with these coordinates to the gradient of the second fundamental form. After some analysis, one sees that these Christoffel symbols can have at best  $L^2$ -integrability in the spherical component. <sup>8</sup> It turns out that this  $L^2$ -integrability for these connection quantities makes it such that the above reduction to the Euclidean estimates just barely fails. In fact, if one has  $L^q$ -integrability for some  $q > 2$ , then this reduction can be recovered. However, this regularity is unattainable, due to the stringent curvature assumptions.

As a result, the authors resorted, in [12], to constructing a fully geometric and tensorial Littlewood-Paley theory via the heat flow. In particular, all the required Besov-type norms and estimates were defined in terms of this theory. The geometric nature of these constructions ultimately circumvented the need for local coordinate

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<sup>7</sup>More specifically, we consider equivariant, or “transported”, coordinate systems, in which we transport a coordinate system on a sphere along the null generators of the cone.

<sup>8</sup>On the other hand, the Christoffel symbols do have  $L^\infty$ -integrability in the “null” component. However, this does not help overcome the difficulty of a lack of spatial integrability.

systems and their associated Christoffel symbols. The price to be paid, though, was a significant amount of added technical baggage to the process, in the forms of numerous elaborate heat flow, Besov, and commutator estimates.

Because of the sheer technical complexity of the above process, other similar low-regularity problems involving evolving geometries remained difficult to approach. As a result, one is interested in simplifying, both lengthwise and technically, the proofs in [13] of the bilinear product estimates. In [22], it was shown that a significant portion of the argument could in fact be made using local coordinate methods. Although this somewhat shortened the proof, due to the difficulties mentioned above, not all of the proofs could be reduced to coordinate analyses. Unfortunately, these “difficult” cases included the null cone analogues of (1.1) and (1.2), which were the toughest estimates to establish in the first place.

A primary goal of this paper is to demonstrate that *all* of these bilinear product estimates in [13, 22], including the analogues of (1.1) and (1.2), do in fact reduce to the corresponding Euclidean versions. Although this could not be done using coordinate considerations, as previously mentioned, we demonstrate here that this can be accomplished, on the other hand, using specially chosen frames. More specifically, we wish to systematically decompose horizontal tensor fields into local scalar quantities via a collection of parallel horizontal frames.<sup>9</sup>

In the motivating null cone settings (e.g., [13, 22]), these parallel frames have additional regularities that were not previously exploited. In fact, one can derive  $L^4$ -integrability in the spherical component for the connection coefficients associated with these frames, which is a strict improvement over frames constructed from local coordinate systems. As mentioned before, this is now sufficient for reducing the geometric bilinear product estimates to their Euclidean analogues.

We mention here that this additional regularity for parallel frames is closely related to the evolutionary covariant derivative. More specifically, the commutation formula between this derivative and other covariant derivatives has nicer algebraic properties than the corresponding commutator using a non-covariant evolutionary derivative.<sup>10</sup> Indeed, in the former commutator, one sees only the curl of the second fundamental form, while in the latter, one sees instead the full gradient. While this “gain” may seem rather negligible at first, it is vastly important in our particular case of null cones with bounded curvature flux.

In this specific setting, the curl of the second fundamental form is further related to the geometry of the spacetime via the well-known Codazzi equations. By taking this into account, one can establish slightly better estimates for the curl of the second fundamental form than for the full gradient. This then leads to the improved  $L^4$ -estimates for the parallel frame connection coefficients.

**Remark.** *In the abstract formulation that we will utilize in most of this paper, we will achieve this improved regularity for parallel frames by postulating corresponding regularity assumptions for this curl of the second fundamental form.*

With the above process, one transforms tensorial quantities to a family of scalar quantities localized to coordinate systems. One can then define Besov-type spaces and norms in the standard fashion with respect to these coordinate systems and an

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<sup>9</sup>More specifically, by “parallel horizontal frames”, we mean that the aforementioned evolutionary, or “vertical”, covariant derivative of these frames vanish.

<sup>10</sup>See formulas (2.2) and (2.3).

associated partition of unity. From here, one can now apply the analogous estimates in Euclidean spaces and then “piece together” the various scalar components into the desired tensorial estimate in our nontrivial geometric setting.

Although the main estimates of this paper are established using coordinate-based Besov norms, some other analyses in [11, 22], as well as in related null cone problems, still require fully geometric Besov norms.<sup>11</sup> Thus, an important issue, separate from the preceding discussions, is that of comparing the coordinate-based Besov norms from our main estimates to analogous geometric norms. We will discuss this problem in our abstract setting at the end of this paper. A part of this argument will follow the same idea as a Besov comparison argument found in [22].

**1.4. Outline of Paper.** We now briefly outline the remainder of this paper.

- In Section 2, we introduce the abstract setting and formalism we will use. In particular, we define the covariant evolutionary derivative and integral operators, and we discuss some of their basic properties. We conclude by describing how this abstract formalism relates to geodesically foliated null cones in Einstein-vacuum spacetimes, that is, the setting of [11].
- In Section 3, we discuss the Euclidean case. We begin by very quickly reviewing some relevant points of classical Littlewood-Paley theory. Afterwards, we proceed to prove the Euclidean versions of the main estimates of this paper. Since some of these estimates were already proved in [13], we elect to keep many of our discussions here brief.
- In Section 4, we discuss in detail the various quantitative assumptions we will impose on our abstract system. We also prove some basic calculus estimates that follow from these assumptions. In the end, we justify these assumptions in terms of our motivating null cone problem by briefly sketching why such conditions would hold in the setting of [11].
- In Section 5, we discuss the process of decomposing horizontal tensor fields to localized scalars via a collection of parallel frames.<sup>12</sup> We then define using classical Littlewood-Paley theory the relevant coordinate-based Besov-type norms with respect to these parallel frames. We conclude this section by stating and proving the main estimates of this paper.
- In Section 6, we construct a geometric tensorial Littlewood-Paley theory based on spectral decompositions of the Böchner Laplacian.<sup>13</sup> We then define invariant Besov-type norms in terms of these geometric Littlewood-Paley operators. Finally, we prove a Besov comparison result – that under certain weak assumptions, described in Section 4, these geometric Besov norms are equivalent to the coordinate-based Besov norms of Section 5.

**1.5. Notations.** Consider a general manifold  $M$  of arbitrary dimension, along with a vector bundle  $\mathcal{V}$  over  $M$ . Given  $p \in M$ , we let  $\mathcal{V}_p$  denote the fiber of  $\mathcal{V}$  at  $p$ . Moreover, we let  $\mathcal{C}^\infty \mathcal{V}$  denote the space of all smooth sections of  $\mathcal{V}$ . Recall that this forms a module over the ring  $C^\infty M$  of all smooth functions on  $M$ .

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<sup>11</sup>For example, there are some Hodge elliptic estimates involving Besov-type norms found in [11, 18, 22] that do not work well with these coordinate-based Besov norms.

<sup>12</sup>Assuming, of course, the conditions described in Section 4.

<sup>13</sup>Equivalently, one could also use the geometric Littlewood-Paley theory developed in [12]. Both theories can be applied in this paper and in the related null cone problems in basically the same manner. Since the spectral theory is easier to rigorously construct, we opt to use that here.

Throughout the paper, we will always let  $r, r_1, r_2$  and  $l, l_1, l_2$  denote non-negative integers. Recall that the prototypical examples of vector bundles over  $M$  are the tensor bundles  $T_l^r M$  on  $M$  of rank  $(r, l)$ .<sup>14</sup> The associated fiber  $(T_l^r M)_p$  of  $T_l^r M$  at any  $p \in M$  is the tensor space on  $M$  of rank  $(r, l)$  at  $p$ . Therefore, the elements of  $\mathcal{C}^\infty T_l^r M$  are the smooth tensor fields of rank  $(r, l)$  on  $M$ .

We will often use standard index notation to describe tensor and tensor fields. Indices, given by lowercase Latin letters, will be with respect to arbitrary or fixed frames and coframes, depending on context. In accordance with Einstein summation notation, repeated indices indicate summations over all allowable index values.

Suppose now  $h \in \mathcal{C}^\infty T_2^0 M$  is a Riemannian metric on  $M$ . We let  $h^{-1} \in \mathcal{C}^\infty T_0^2 M$  denote the metric dual of  $h$ . As usual, within index notation,  $h^{-1}$  is written as simply  $h$ , but with superscript indices.<sup>15</sup> Recall that  $h$  and  $h^{-1}$  define pointwise tensorial inner products and norms on  $M$ : for any  $F, G \in \mathcal{C}^\infty T_l^r M$ , we define

$$\langle F, G \rangle_{(h)} = h_{a_1 b_1} \dots h_{a_r b_r} h^{c_1 d_1} \dots h^{c_s d_s} F^{a_1 \dots a_r}_{c_1 \dots c_s} G^{b_1 \dots b_r}_{d_1 \dots d_s} \in \mathcal{C}^\infty M.$$

This is precisely the bundle metric on  $T_l^r M$  induced by  $h$ . We also define

$$|F|_{(h)} = \langle F, F \rangle_{(h)}^{\frac{1}{2}},$$

i.e., the pointwise tensor norms with respect to  $h$ .<sup>16</sup> We can then use these pointwise norms to define standard integral norms, again with respect to  $h$ :

$$\|F\|_{L_x^q(h)} = \left( \int_M |F|_{(h)}^q \right)^{\frac{1}{q}}, \quad \|F\|_{L_x^\infty(h)} = \sup_{\omega \in M} |F|_{(h)}|_\omega, \quad q \in [1, \infty),$$

where the above integral is with respect to the volume measure induced by  $h$ . Finally, following standard conventions, we let  $\nabla^{(h)}$  and  $\Delta^{(h)}$  denote the Levi-Civita connection and the Böchner Laplacian with respect to  $h$ , respectively.

As is standard, given nonnegative real numbers  $A$  and  $B$ , we write

$$A \lesssim B$$

to mean that  $A \leq CB$  for some positive universal constant  $C$ , and we write

$$A \simeq B$$

to mean both  $A \lesssim B$  and  $B \lesssim A$ . More generally, given nonnegative real numbers  $c, c_1, \dots, c_n$  and  $C_1, \dots, C_m$ , we will use the notation

$$A \lesssim_{c_1, \dots, c_m}^c B$$

to mean that  $A \leq e^{C'} C B$  for some  $C, C' > 0$  depending on  $c_1, \dots, c_m$ . Similarly,

$$A \simeq_{c_1, \dots, c_m}^c B$$

means that both of the following statements hold:

$$A \lesssim_{c_1, \dots, c_m}^c B, \quad B \lesssim_{c_1, \dots, c_m}^c A.$$

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<sup>14</sup>Here,  $r$  is the contravariant rank, and  $l$  is the covariant rank.

<sup>15</sup>In other words, the components  $h^{ab}$  comprise the inverse matrix of the  $h_{ab}$ 's.

<sup>16</sup>In the scalar case  $r = l = 0$ , the inner product  $\langle \cdot, \cdot \rangle$  is simply multiplication of functions, and the norm  $|\cdot|$  is the absolute value. In particular, these are independent of  $h$ .



## 2. GENERAL SURFACE FOLIATIONS

In this section, we will discuss the abstract setting of our analyses. Let  $\mathcal{S}$  denote an arbitrary 2-dimensional manifold, and consider the foliation

$$\mathcal{N} = [0, \delta) \times \mathcal{S}, \quad \delta > 0.$$

Define  $t$  to be the natural projection onto the first component:

$$t : \mathcal{N} \rightarrow [0, \delta), \quad t(v, q) = v.$$

Throughout, we will let  $v$  denote an arbitrary element on the interval  $[0, \delta)$ . Given such a  $v$ , we let  $\mathcal{S}_v$  denote the associated level set of  $t$ :

$$\mathcal{S}_v = t^{-1}(v) = \{v\} \times \mathcal{S}.$$

**Remark.** *In fact, one could consider only the case  $\delta = 1$ , as the case for general  $\delta$  can be obtained from this by a standard rescaling argument for  $t$ .*

Although we will work only on the 3-manifold with boundary  $\mathcal{N}$ , we will always implicitly assume that all our objects can be smoothly extended beyond the lower boundary  $\mathcal{S}_0$ . In other words, our full setting, on which all our objects of analysis are defined, is the extended foliation  $\mathcal{N}' = (-\delta', \delta) \times \mathcal{S}$ , for some  $\delta' > 0$ .

**2.1. Covariant Structures.** Consider the trivial diffeomorphism

$$\Xi_v : \mathcal{S}_v \leftrightarrow \mathcal{S}, \quad \Xi_v(v, \omega) = \omega,$$

which identifies  $\mathcal{S}_v$  with  $\mathcal{S}$ . This map can be canonically extended to tensors and tensor fields via pullbacks and push-forwards, i.e.,

$$\Xi_v^* : \mathcal{C}^\infty T_l^r \mathcal{S}_v \leftrightarrow \mathcal{C}^\infty T_l^r \mathcal{S}.$$

We will use these identifications repeatedly in our basic constructions.

The first task is to define objects that represent, roughly, a family of corresponding objects on the  $\mathcal{S}_v$ 's varying smoothly with respect to  $v$ . For this purpose, we can naturally construct a vector bundle  $\underline{T}_l^r \mathcal{N}$  over  $\mathcal{N}$ , with fibers

$$(\underline{T}_l^r \mathcal{N})_p = (T_l^r \mathcal{S}_v)_p, \quad p \in (v, \omega) \in \mathcal{N}.$$

We call  $\underline{T}_l^r \mathcal{N}$  the *horizontal tensor bundle* of rank  $(r, l)$ . In particular,  $\underline{T}_0^0 \mathcal{N}$  can be naturally identified with the space  $C^\infty \mathcal{N}$  of smooth functions on  $\mathcal{N}$ . We will often make this identification without further mention.<sup>17</sup>

An element  $A \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$  is called a *horizontal tensor field*. In general, given  $v$ , we will let  $A[v] \in \mathcal{C}^\infty T_l^r \mathcal{S}$  denote the tensor field on  $\mathcal{S}$  that represents the restriction of  $A$  to  $\mathcal{S}_v$ .<sup>18</sup> Alternately, we can think of  $A$  as the family  $A[v] \in \mathcal{C}_0^\infty T_l^r \mathcal{S}$  of tensor fields on  $\mathcal{S}$  which also varies smoothly with respect to  $v$ .

The next structure to impose on  $\mathcal{N}$  is a *horizontal metric*  $\gamma \in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}$ . More specifically, we stipulate that  $\gamma[v]$  is a Riemannian metric on  $\mathcal{S}$  for each  $v$ . We also let  $\gamma^{-1} \in \mathcal{C}^\infty \underline{T}_0^2 \mathcal{N}$  denote the dual to  $\gamma$ , that is,  $\gamma^{-1}[v]$  is the dual  $(\gamma[v])^{-1}$  to  $\gamma[v]$  for each  $v$ . In addition to  $\gamma$ , we also assume an orientation on  $\mathcal{S}$ . This induces a *horizontal volume form*  $\epsilon \in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}$ , such that the restriction  $\epsilon[v]$  represents the volume form of  $\mathcal{S}$  associated with  $\gamma[v]$  and the orientation of  $\mathcal{S}$ .<sup>19</sup>

<sup>17</sup>Recall that we always assume elements of  $\mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$  to be smoothly extendible beyond  $\mathcal{S}_0$ .

<sup>18</sup>More specifically, one can define  $A[v] = \Xi_v^*(A|_{\mathcal{S}_v})$ .

<sup>19</sup>As we will only be working with  $\mathbb{R}^2$  and compact manifolds in practice, our assumption of orientation and volume form creates no loss of generality.

In general, families of objects defined on  $\mathcal{S}$  that are parametrized by  $v$  can be aggregated into corresponding “horizontal” objects on  $\mathcal{N}$ . We already saw three examples of this in the definitions of  $\gamma$ ,  $\gamma^{-1}$ , and  $\epsilon$ . We now list the remaining common examples we will reference throughout the paper.

First, the pointwise inner products with respect to the  $\gamma[v]$ ’s lift to a corresponding inner product on horizontal tensor fields, with respect to  $\gamma$ :

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{(\gamma)} : \mathcal{C}^\infty \underline{T}_l^r \mathcal{N} \times \mathcal{C}^\infty \underline{T}_l^r \mathcal{N} \rightarrow \mathcal{C}^\infty \mathcal{N}, \quad \langle \Psi, \Phi \rangle[v] = \langle \Psi[v], \Phi[v] \rangle_{(\gamma[v])}.$$

The horizontal tensor norm  $|\cdot| = |\cdot|_{(\gamma)}$  is similarly defined from  $|\cdot|_{\gamma[v]}$ .

Next, we can in a similar manner naturally define tensor products of horizontal tensor fields. Given  $\Psi_i \in \mathcal{C}^\infty \underline{T}_{l_i}^{r_i}$ , where  $i \in \{1, 2\}$ , we define

$$\Psi_1 \otimes \Psi_2 \in \mathcal{C}^\infty \underline{T}_{l_1+l_2}^{r_1+r_2}, \quad (\Psi_1 \otimes \Psi_2)[v] = \Psi_1[v] \otimes \Psi_2[v].$$

In other words, on each  $\mathcal{S}_v$ , the product  $\Psi_1 \otimes \Psi_2$  is defined to be the tensor product of the respective restrictions of  $\Psi_1$  and  $\Psi_2$  to  $\mathcal{S}_v$ .

Moreover, the Levi-Civita connections  $\nabla^{(\gamma[v])}$  with respect to the  $\gamma[v]$ ’s can be aggregated into a single *horizontal covariant differential operator*

$$\nabla = \nabla^{(\gamma)} : \mathcal{C}^\infty \underline{T}_l^r \mathcal{N} \rightarrow \mathcal{C}^\infty \underline{T}_{l+1}^r \mathcal{N}.$$

More specifically, if  $A \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$  and  $X \in \mathcal{C}^\infty \underline{T}_0^1 \mathcal{N}$ , then

$$(\nabla_X A)[v] = \nabla_{X[v]}^{(\gamma[v])}(A[v]).$$

Higher order covariant differentials can then be defined iteratively:

$$\nabla^k : \mathcal{C}^\infty \underline{T}_l^r \mathcal{N} \rightarrow \mathcal{C}^\infty \underline{T}_{l+k}^r \mathcal{N}, \quad \nabla^k = \nabla \nabla^{k-1}, \quad k > 1.$$

We can also define the horizontal (Böchner) Laplacian with respect to  $\gamma$ :

$$\Delta = \Delta^{(\gamma)} : \mathcal{C}^\infty \underline{T}_l^r \mathcal{N} \rightarrow \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}, \quad (\Delta \Psi)[v] = \Delta^{(\gamma[v])}(\Psi[v]).$$

Basically, the restriction to  $\mathcal{S}_v$  of  $\Delta$  is the Laplacian on  $\mathcal{S}$  with respect to  $\gamma[v]$ .

**Remark.** For convenience, whenever unambiguous, we will omit the dependence of various quantities on the horizontal metric and orientation. For instance, as defined above,  $|\cdot|$  and  $\nabla$  refer to  $|\cdot|_{(\gamma)}$  and  $\nabla^{(\gamma)}$ , respectively.

If  $U$  is an open subset of  $\mathcal{S}$ , then we can consider the localized foliation<sup>20</sup>

$$\mathcal{N}_U = [0, \delta) \times U.$$

Since  $U$  is a submanifold of  $\mathcal{S}$  of the same dimension, then we can treat  $\mathcal{N}_U$  using the exact same formalisms as we do for  $\mathcal{N}$ . All of the geometric quantities we have defined on  $\mathcal{N}$  have direct analogues on  $\mathcal{N}_U$  which are obtained directly via restriction (e.g.,  $\gamma$  and  $\nabla$ ). These localized foliations will be generally useful for dealing with local coordinate systems and local frames.

Relations involving horizontal tensors are often more easily described using index notations. We will essentially use the same indexing conventions as one would use for tensors on  $\mathcal{S}$  or the  $\mathcal{S}_v$ ’s. We will use lowercase Latin indices to denote components of a horizontal tensor field, with repeated indices indicating summations.

To be a bit more specific, suppose  $U \subseteq \mathcal{S}$  is open, and let

$$e_1, e_2 \in \mathcal{C}^\infty \underline{T}_0^1 \mathcal{N}_U, \quad e_*^1, e_*^2 \in \mathcal{C}^\infty \underline{T}_1^0 \mathcal{N}_U$$

<sup>20</sup>We still assume in this localized case that all objects are smoothly extendible past  $\mathcal{S}_0$ .

denote a local horizontal frame and associated coframe. We can index horizontal tensor fields with respect to these frames: for example, if  $\Psi \in \mathcal{C}^\infty \underline{T}_1^1$ , then

$$\Psi_b^a = \Psi(e_*^a, e_b) \in C^\infty \mathcal{N}_U.$$

Note that the chosen frame and coframe is allowed to change as  $v$  changes.

**Remark.** In particular, if we index on  $\mathcal{N}$  as above, and if we index on each  $\mathcal{S}_v \simeq \mathcal{S}$  with respect to the  $e_a[v]$ 's and  $e_*^b[v]$ 's, then for any  $\Psi \in \mathcal{C}^\infty \underline{T}_1^1 \mathcal{N}$ , we have

$$\Psi_b^a[v] = (\Psi[v])_b^a.$$

This extends directly to horizontal tensor fields of any rank.

**Remark.** To remain consistent with standard index notation conventions, the components of  $\gamma^{-1}$  will be denoted  $\gamma^{ab}$  rather than  $(\gamma^{-1})^{ab}$ .

**2.2. Evolution.** The identifications  $\Xi_v$  can be utilized to define a *vertical Lie derivative* of horizontal tensor fields. Given  $A \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , we can define

$$\mathfrak{L}_t A[v] = \lim_{v' \rightarrow v} \frac{A[v'] - A[v]}{v' - v} \in \mathcal{C}^\infty T_l^r \mathcal{S}.$$

Like for previous definitions, the quantities  $\mathfrak{L}_t A[v]$  aggregate to define our desired Lie derivative  $\mathfrak{L}_t A \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ . Heuristically, this  $\mathfrak{L}_t A$  measures how  $A$  evolves as  $t$  increases, with respect to the diffeomorphisms  $\Xi_v$ .

**Remark.** Note that the operator  $\mathfrak{L}_t$  is independent of  $\gamma$  and  $\epsilon$ .

A horizontal field  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$  is called *equivariant* iff  $\mathfrak{L}_t \Psi$  vanishes everywhere. Equivalently,  $\Psi$  is equivariant if and only if the  $\Psi[v]$ 's remain constant with respect to  $v$ . Given any  $F \in \mathcal{C}^\infty T_l^r \mathcal{S}$ , we let  $\mathfrak{e}F \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$  denote the *equivariant transport* of  $F$ , i.e., the unique equivariant field satisfying  $\mathfrak{e}F[v] = F$  for every  $v$ .

For example, if  $(U, \tilde{\varphi}) = (U; \tilde{x}^1, \tilde{x}^2)$  is a local coordinate system on  $\mathcal{S}$ , then we can transport each coordinate  $\tilde{x}^i$  to  $\mathcal{N}_U$  equivariantly:

$$x^i = \mathfrak{e}\tilde{x}^i \in C^\infty \mathcal{N}_U, \quad i \in \{1, 2\}.$$

This defines a pair of equivariant functions on  $\mathcal{N}_U$  which form coordinate systems on each timeslice  $U_v = \{v\} \times U$ . Furthermore, if  $\tilde{\partial}_1$  and  $\tilde{\partial}_2$  are the coordinate vector fields associated to  $\tilde{x}^1$  and  $\tilde{x}^2$ , then their equivariant transports

$$\partial_i = \mathfrak{e}\tilde{\partial}_i \in \mathcal{C}^\infty \underline{T}_0^1 \mathcal{N}_U, \quad i \in \{1, 2\}$$

form the coordinate vector fields for the  $x^i$ 's on each  $U_v$ .

We define the *second fundamental form*, with respect to  $\gamma$ , to be

$$k = k^{(\gamma)} = \frac{1}{2} \mathfrak{L}_t \gamma \in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}.$$

This describes the evolution of the metrics  $\gamma[v]$  as  $v$  increases. Of particular importance will be the  $\gamma$ -trace of  $k$ , called the *mean curvature*, or *expansion*, of  $k$ :

$$\text{tr } k = \text{tr}^{(\gamma)} k^{(\gamma)} = \gamma^{ab} k_{ab}^{(\gamma)} \in C^\infty \mathcal{N}.$$

Elementary computations yield the following basic identities:

$$(2.1) \quad \frac{1}{2} (\mathfrak{L}_t \gamma^{-1})^{ab} = -\gamma^{ac} \gamma^{bd} k_{cd}, \quad (\mathfrak{L}_t \epsilon)_{ab} = (\text{tr } k) \epsilon_{ab}.$$

The second identity in (2.1) justifies the name “expansion” given to  $\text{tr } k$ .

A direct calculation also yields the following commutator identity for  $\mathfrak{L}_t$  and  $\nabla$ .

**Proposition 2.1.** *If  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , then*

$$(2.2) \quad [\mathfrak{L}_t, \nabla_a] \Psi_{u_1 \dots u_l}^{v_1 \dots v_r} = - \sum_{i=1}^l \gamma^{cd} (\nabla_a k_{u_i c} + \nabla_{u_i} k_{ac} - \nabla_c k_{a u_i}) \Psi_{u_1 \hat{d}_i u_l}^{v_1 \dots v_r} \\ + \sum_{j=1}^r \gamma^{cv_j} (\nabla_a k_{dc} + \nabla_d k_{ac} - \nabla_c k_{ad}) \Psi_{u_1 \dots u_l}^{v_1 \hat{d}_j v_r}.$$

Here,  $u_1 \hat{d}_i u_l$  denotes the set of indices  $u_1 \dots u_l$ , but with  $u_i$  replaced by  $d$ . The upper index notation  $v_1 \hat{d}_j v_r$  is defined analogously.

We can now use  $\mathfrak{L}_t$  and  $k$  to define a corresponding *covariant* derivative along the  $t$ -direction. Given  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , we define  $\nabla_t \Psi = \nabla_t^{(\gamma)} \Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$  by

$$\nabla_t \Psi_{u_1 \dots u_l}^{v_1 \dots v_r} = \mathfrak{L}_t \Psi_{u_1 \dots u_l}^{v_1 \dots v_r} - \sum_{i=1}^l \gamma^{cd} k_{u_i c} \Psi_{u_1 \hat{d}_i u_l}^{v_1 \dots v_r} + \sum_{j=1}^r \gamma^{cv_j} k_{cd} \Psi_{u_1 \dots u_l}^{v_1 \hat{d}_j v_r},$$

where we use the same multi-index conventions as in Proposition 2.1. In particular, note that  $\nabla_t$  and  $\mathfrak{L}_t$  coincide in the case of scalar fields.

Next, we define the quantity

$$\mathfrak{C} = \mathfrak{C}^{(\gamma)} \in \mathcal{C}^\infty \underline{T}_3^0 \mathcal{N}, \quad \mathfrak{C}_{abc} = \nabla_b k_{ac} - \nabla_c k_{ab},$$

representing the “curl” of  $k$ . With this, we can state the commutation formula for  $\nabla_t$  and  $\nabla$ , which is a result of basic computations that we leave to the reader.

**Proposition 2.2.** *If  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , then*

$$(2.3) \quad [\nabla_t, \nabla_a] \Psi_{u_1 \dots u_l}^{v_1 \dots v_r} = -\gamma^{cd} k_{ac} \nabla_d \Psi_{u_1 \dots u_l}^{v_1 \dots v_r} - \sum_{i=1}^l \gamma^{cd} \mathfrak{C}_{a u_i c} \Psi_{u_1 \hat{d}_i u_l}^{v_1 \dots v_r} \\ + \sum_{j=1}^r \gamma^{cv_j} \mathfrak{C}_{ad c} \Psi_{u_1 \dots u_l}^{v_1 \hat{d}_j v_r}.$$

The contrast between (2.2) and (2.3) will play a fundamental role in the analysis. Observe that in (2.2), we have terms on the right-hand side of the form  $\nabla k \otimes \Psi$ . In (2.3), however, the gradient  $\nabla k$  is replaced by  $\mathfrak{C}$ , which can possess additional structure and regularity in certain situations.<sup>21</sup>

We say that  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$  is *t-parallel* (with respect to  $\gamma$ ), iff  $\nabla_t \Psi \equiv 0$ .<sup>22</sup> Most importantly, one can see that  $\gamma$ ,  $\gamma^{-1}$ , and  $\epsilon$  are all *t-parallel*. We can interpret this as  $\nabla_t$  being “compatible” with both the horizontal metric and volume form.<sup>23</sup> Given any  $F \in \mathcal{C}^\infty T_l^r \mathcal{S}$ , then one can solve for a unique *t-parallel transport*

$$\mathfrak{p}F = \mathfrak{p}^{(\gamma)} F \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N},$$

such that  $\mathfrak{p}F$  is *t-parallel* and  $(\mathfrak{p}F)[0] = F$ . This is analogous with establishing the usual parallel transport in pseudo-Riemannian geometry.

<sup>21</sup>For example, if  $\gamma$  represents the metrics induced from a larger Lorentz manifold  $(M, g)$ , then  $\mathfrak{C}$  is present in the Codazzi equations relating the curvature of  $M$  to the geometries of the  $\mathcal{S}_v$ 's.

<sup>22</sup>Such fields were sometimes called *Fermi-transported*, e.g., [11].

<sup>23</sup>In particular, the Levi-Civita connections  $\nabla$  along with the operator  $\nabla_t$  combine to form vector bundle connections for the horizontal bundles  $\underline{T}_l^r \mathcal{N}$ . In addition, these connections are compatible with the bundle metrics  $\langle \cdot, \cdot \rangle$  for the  $\underline{T}_l^r \mathcal{N}$ 's.

Note that if  $\Psi \in \mathcal{C}^\infty \underline{T}_t^r \mathcal{N}$  is  $t$ -parallel, then its tensor norm is preserved:<sup>24</sup>

$$|\Psi|_{(v,\omega)} = |\Psi|_{(0,\omega)}.$$

In particular, we have the relation

$$|\mathfrak{p}F|_{(v,\omega)} = |F[0]|_{(\gamma[0])}|_\omega, \quad F \in \mathcal{C}^\infty T_t^r \mathcal{S}.$$

Finally, note that since  $\mathfrak{L}_t$  and  $\nabla_t$  coincide for scalar fields, then the equivariant and  $t$ -parallel propagators  $\mathfrak{e}$  and  $\mathfrak{p}$  coincide in the scalar setting.

**2.3. Covariant Integration.** The next task is to define a notion of an evolutionary horizontal covariant integral. The simplest method to construct this is via  $t$ -parallel frames and coframes. If  $U$  is an open neighborhood of  $\mathcal{S}$ , and if  $\tilde{e}_1, \tilde{e}_2 \in \mathcal{C}^\infty T_0^1 U$  forms a local frame on  $U$ , then the  $t$ -parallel transports

$$e_i = \mathfrak{p}\tilde{e}_i \in \mathcal{C}^\infty \underline{T}_0^1 \mathcal{N}_U, \quad i \in \{1, 2\}$$

form a local frame on each of the  $U_v$ 's. Moreover, if  $\tilde{e}_1$  and  $\tilde{e}_2$  are orthonormal, then  $e_1$  and  $e_2$  remain orthonormal as well. A completely analogous construction can be made using coframes instead of frames.

Given  $\Psi \in \mathcal{C}^\infty \underline{T}_t^r \mathcal{N}$ , we define the *covariant  $t$ -integral*  $\mathfrak{T}\Psi \in \mathcal{C}^\infty \underline{T}_t^r \mathcal{N}$  as follows. Indexing with respect to an arbitrary local  $t$ -parallel frame and coframe, we set

$$(\mathfrak{T}\Psi)_{u_1 \dots u_l}^{v_1 \dots v_r} = \int_0^v (\Psi_{u_1 \dots u_l}^{v_1 \dots v_r})|_{(w,\omega)} dw, \quad (v, \omega) \in \mathcal{N}.$$

Here, the right-hand side represents the standard integral over the scalar  $\Psi_{u_1 \dots u_l}^{v_1 \dots v_r}$ . Note that if  $\phi \in \mathcal{C}^\infty \mathcal{N}$ , then  $\mathfrak{T}\phi$  is in fact the usual integral:

$$\mathfrak{T}\phi|_{(v,\omega)} = \int_0^v \phi|_{(w,\omega)} dw, \quad (v, \omega) \in \mathcal{N}.$$

As a result, one can view  $\mathfrak{T}$  as a covariant extension of the standard integral.

From the fundamental theorem of calculus, we have

$$\nabla_t \mathfrak{T}\Psi = \Psi, \quad \mathfrak{T}\nabla_t \Psi = \Psi - \mathfrak{p}(\Psi[0]), \quad \Psi \in \mathcal{C}^\infty \underline{T}_t^r \mathcal{N}.$$

In particular, if  $\Psi[0]$  vanishes entirely, then  $\mathfrak{T}$  acts as a formal inverse to  $\nabla_t$ . In fact, when we “integrate” covariant evolution equations involving horizontal tensor fields, we are actually applying these covariant integral operators  $\mathfrak{T}$ .

**Remark.** Note that  $(\mathfrak{T}\Psi)[0]$  vanishes everywhere on  $\mathcal{S}$  by definition.

**Proposition 2.3.** If  $\Psi \in \mathcal{C}^\infty \underline{T}_t^r \mathcal{N}$ , then

$$(2.4) \quad \begin{aligned} |\mathfrak{T}\Psi| &\leq \mathfrak{T}|\Psi|, \\ |\Psi| &\leq |\mathfrak{p}(\Psi[0])| + |\mathfrak{T}\nabla_t \Psi|. \end{aligned}$$

*Proof.* Indexing with respect to an orthonormal  $t$ -parallel frame, we have

$$|\mathfrak{T}\Psi|^2|_{(v,\omega)} = \sum_{\substack{u_1 \dots u_l \\ v_1 \dots v_r}} (\mathfrak{T}\Psi_{u_1 \dots u_l}^{v_1 \dots v_r})^2|_{(v,\omega)} = \sum_{\substack{u_1 \dots u_l \\ v_1 \dots v_r}} \left( \int_0^v \Psi_{u_1 \dots u_l}^{v_1 \dots v_r}|_{(w,\omega)} dw \right)^2$$

for any  $(v, \omega) \in \mathcal{N}$ . By the integral Minkowski inequality, we obtain

$$|\mathfrak{T}\Psi|_{(v,\omega)} \leq \int_0^v \left[ \sum_{\substack{u_1 \dots u_l \\ v_1 \dots v_r}} (\Psi_{u_1 \dots u_l}^{v_1 \dots v_r})^2|_{(w,\omega)} \right]^{1/2} dw = \int_0^v |\Psi|_{(w,\omega)} dw,$$

<sup>24</sup>Moreover, the  $\gamma$ -inner product is preserved in the  $t$ -direction as well.

which proves the first inequality of (2.4). Using the same frame, we also have

$$\Psi_{u_1 \dots u_l}^{v_1 \dots v_r}|_{(v, \omega)} = \Psi_{u_1 \dots u_l}^{v_1 \dots v_r}|_{(0, \omega)} + (\mathfrak{T} \nabla_t \Psi)_{u_1 \dots u_l}^{v_1 \dots v_r}|_{(v, \omega)}.$$

Squaring the above and summing over all the indices yields

$$|\Psi|^2|_{(v, \omega)} = \sum_{\substack{u_1 \dots u_l \\ v_1 \dots v_r}} \left[ \Psi_{u_1 \dots u_l}^{v_1 \dots v_r}|_{(0, \omega)} + (\mathfrak{T} \nabla_t \Psi)_{u_1 \dots u_l}^{v_1 \dots v_r} \right]^2.$$

Applying Minkowski's inequality yields the second part of (2.4).  $\square$

Finally, we briefly discuss commutations involving  $\mathfrak{T}$ . Consider an operator

$$L : \mathcal{C}^\infty \underline{T}_{l_1}^{r_1} \mathcal{N} \rightarrow \mathcal{C}^\infty \underline{T}_{l_2}^{r_2} \mathcal{N}.$$

By the relations between  $\mathfrak{T}$  and  $\nabla_t$ , we have the formula

$$[\mathfrak{T}, L]\Psi = -\mathfrak{T}[\nabla_t, L]\mathfrak{T}\Psi - \mathfrak{p}(L\mathfrak{T}\Psi[0]).$$

If the last term on the right-hand side vanishes, then

$$[\mathfrak{T}, L]\Psi = -\mathfrak{T}[\nabla_t, L]\mathfrak{T}\Psi.$$

One important consequence of this is that  $\mathfrak{T}$  commutes with any such  $L$  that commutes with  $\nabla_t$ , as long as  $L\mathfrak{T}\Psi$  vanishes on  $\mathcal{S}_0$ . In particular, since  $\nabla_t$  commutes with all contractions, metric contractions, and volume form contractions, then  $\mathfrak{T}$  commutes with these operations as well. Another important case is when  $L = \nabla$ , as we can now use (2.3) to commute  $\nabla$  and  $\mathfrak{T}$ .

**Proposition 2.4.** *If  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , then*

$$(2.5) \quad \begin{aligned} [\mathfrak{T}, \nabla_a] \Psi_{u_1 \dots u_l}^{v_1 \dots v_r} &= \gamma^{cd} \mathfrak{T}(k_{ac} \nabla_d \mathfrak{T} \Psi_{u_1 \dots u_l}^{v_1 \dots v_r}) + \sum_{i=1}^l \gamma^{cd} \mathfrak{T}(\mathfrak{C}_{au_i c} \mathfrak{T} \Psi_{u_1 \dots \hat{u}_i \dots u_l}^{v_1 \dots v_r}) \\ &\quad - \sum_{j=1}^r \gamma^{cv_j} \mathfrak{T}(\mathfrak{C}_{adc} \mathfrak{T} \Psi_{u_1 \dots u_l}^{v_1 \dots \hat{v}_j \dots v_r}). \end{aligned}$$

**2.4. Integral Norms.** Integrating the second identity of (2.1), we have

$$\mathfrak{L}_t \{ \exp[-\mathfrak{T}(\text{tr } k)] \cdot \epsilon \} \equiv 0.$$

This equivariance implies the following identity:

$$\{ \exp[-\mathfrak{T}(\text{tr } k)] \cdot \epsilon \} [v] = \epsilon[0].$$

Define now the associated *Jacobian* with respect to  $\epsilon$  to be the factor

$$\mathcal{J} = \mathcal{J}^{(\epsilon)} = \exp \mathfrak{T}(\text{tr } k) \in C^\infty \mathcal{N}.$$

$\mathcal{J}$  acts as a “change of measure” quantity, as it satisfies

$$(2.6) \quad \epsilon[v] = \mathcal{J}[v] \cdot \epsilon[0], \quad \nabla_t \mathcal{J} = \text{tr } k \cdot \mathcal{J}.$$

We also note the identity

$$\int_{\mathcal{S}} \phi \cdot d\epsilon[v] = \int_{\mathcal{S}} \phi \mathcal{J}[v] \cdot d\epsilon[0], \quad \phi \in C^\infty \mathcal{S},$$

which is a direct consequence of (2.6).

Next, given  $q \in [1, \infty]$  and  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , we define

$$\|\Psi[v]\|_{L_x^q} = \|\Psi[v]\|_{L_x^q(\gamma)} = \|\Psi[v]\|_{L_x^q(\gamma[v])},$$

whenever such an integral is finite. The main point here is that we implicitly assume these integral norms of  $\Psi[v]$  are to be with respect to  $\gamma[v]$ .

In addition, we define the following iterated integral norms:

- If  $p \in [1, \infty)$  and  $q \in [1, \infty]$ , then we define

$$\begin{aligned} \|\Psi\|_{L_{t,x}^{p,q}} &= \|\Psi\|_{L_{t,x}^{p,q}(\gamma)} = \left( \int_0^\delta \|\Psi[v]\|_{L_x^q}^r dv \right)^{\frac{1}{r}}, \\ \|\Psi\|_{L_{t,x}^{\infty,q}} &= \|\Psi\|_{L_{t,x}^{\infty,q}(\gamma)} = \sup_{0 \leq v < \delta} \|\Psi[v]\|_{L_x^q}. \end{aligned}$$

- We can also reverse the order of integration. Given  $p, q \in [1, \infty)$ , we define

$$\begin{aligned} \|\Psi\|_{L_{x,t}^{q,p}} &= \|\Psi\|_{L_{x,t}^{q,p}(\gamma)} = \left[ \int_{\mathcal{S}} \left( \int_0^\delta |\Psi|^p \mathcal{J}^{\frac{p}{q}} \Big|_{(v,\omega)} dv \right)^{\frac{q}{p}} d\epsilon[0]_\omega \right]^{\frac{1}{q}}, \\ \|\Psi\|_{L_{x,t}^{q,\infty}} &= \|\Psi\|_{L_{x,t}^{q,\infty}(\gamma)} = \left[ \int_{\mathcal{S}} \left( \sup_{0 \leq v < \delta} |\Psi| \mathcal{J}^{\frac{1}{q}} \Big|_{(v,\omega)} \right)^q d\epsilon[0]_\omega \right]^{\frac{1}{q}}. \end{aligned}$$

Furthermore, when  $q = \infty$ , we define

$$\begin{aligned} \|\Psi\|_{L_{x,t}^{\infty,p}} &= \|\Psi\|_{L_{x,t}^{\infty,p}(\gamma)} = \sup_{\omega \in \mathcal{S}} \left( \int_0^\delta |\Psi|^p \Big|_{(v,\omega)} dv \right)^{\frac{1}{p}}, \\ \|\Psi\|_{L_{x,t}^{\infty,\infty}} &= \|\Psi\|_{L_{x,t}^{\infty,\infty}(\gamma)} = \sup_{\omega \in \mathcal{S}} \sup_{0 \leq v < \delta} |\Psi| \Big|_{(v,\omega)}. \end{aligned}$$

- For convenience, we also define the iterated Sobolev norm

$$\|\Psi\|_{H_{t,x}^1} = \delta \|\nabla_t \Psi\|_{L_{t,x}^{2,2}} + \|\nabla \Psi\|_{L_{t,x}^{2,2}} + \|\Psi\|_{L_{t,x}^{2,2}}.$$

It is clear that the  $L_x^q$ -norms, as well as the iterated  $L_{t,x}^{p,q}$ - and  $L_{x,t}^{q,p}$ -norms, satisfy the usual Hölder, Minkowski, and integral Minkowski inequalities.

**2.5. Regular Vacuum Null Cones.** We now discuss how our abstract formalism applies to our main setting of interest: null cones on Lorentzian manifolds. For simplicity, we only discuss geodesically foliated null cones in Einstein-vacuum spacetimes; this is the setting found in [11]. Other variants of such null cone problems (e.g., other foliations of null cones, null cones beginning from a vertex, non-vacuum spacetimes) can also be described using the formalisms of this section, although the specifics will differ very slightly from the development here.

Let  $(M, g)$  denote a four-dimensional time-oriented Lorentzian manifold. We assume  $(M, g)$  is *Einstein-vacuum*, i.e., that it is Ricci flat.

- Fix a compact two-dimensional submanifold  $\mathcal{S}$  of  $M$  diffeomorphic to  $\mathbb{S}^2$ .
- We smoothly assign to each  $p \in \mathcal{S}$  a future (or alternately, past) null vector  $\ell_p$  at  $p$  which is in addition orthogonal to  $\mathcal{S}$ .
- For each  $p \in \mathcal{S}$ , we let  $\lambda_p$  denote the future (or past) null geodesic satisfying the initial conditions  $\lambda_p(0) = p$  and  $\lambda_p'(0) = \ell_p$ .
- Let the (truncated) *null cone*  $\mathcal{N}$  denote a smooth portion of the null hypersurface traced out by the congruence  $\{\lambda_p | p \in \mathcal{S}\}$  of null geodesics.
- Let  $t : \mathcal{N} \rightarrow \mathbb{R}$  denote the *affine parameter*, which maps  $\lambda_p(v) \in \mathcal{N}$ , where  $p \in \mathcal{S}$  and  $v \in \mathbb{R}$ , to the number  $v$ . Note that  $t$  is smooth and well-defined as long as  $\mathcal{N}$  avoids both null conjugate and null cut locus points.

Identifying  $\mathcal{S}$  with  $\mathbb{S}^2$ , then we can consider  $\mathcal{N}$  as

$$\mathcal{N} \simeq [0, \delta) \times \mathbb{S}^2, \quad \delta > 0.$$

We also define the following on  $\mathcal{N}$ :

- We define the vector field  $L$  on  $\mathcal{N}$  to be the tangent vector fields of the  $\lambda_p$ 's. By definition,  $L$  is geodesic, and  $Lt \equiv 1$  everywhere. This implies that the level sets  $\mathcal{S}_v$  of  $t$  are in fact Riemannian submanifolds of  $\mathcal{N}$  and  $M$ .
- Let the horizontal metric  $\gamma \in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}$  be defined as the metrics on the  $\mathcal{S}_v$ 's induced from the spacetime metric  $g$ .
- Let  $\underline{L}$  denote the conjugate null vector field on  $\mathcal{N}$ , which is orthogonal to every  $\mathcal{S}_v$  and satisfies  $g(L, \underline{L}) \equiv -2$ . Note that  $\underline{L}$  is transverse to  $\mathcal{N}$ : for any  $q \in \mathcal{N}$ , the vector  $\underline{L}|_q$  is a tangent vector for  $M$ , but not  $\mathcal{N}$ .

Next, we define the *Ricci coefficients*, which are horizontal connection quantities that describe the derivatives of  $L$  and  $\underline{L}$  in the directions tangent to  $\mathcal{N}$ .

- Define 2-tensors  $\chi, \underline{\chi} \in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}$  by

$$\chi(X, Y) = g(D_X L, Y), \quad \underline{\chi}(X, Y) = g(D_X \underline{L}, Y), \quad X, Y \in \mathcal{C}^\infty \underline{T}_0^1 \mathcal{N},$$

where  $D$  is the restriction of the spacetime Levi-Civita connection to  $\mathcal{N}$ .

- Define  $\zeta \in \mathcal{C}^\infty \underline{T}_1^0 \mathcal{N}$  by

$$\zeta(X) = \frac{1}{2} g(D_X L, \underline{L}), \quad X \in \mathcal{C}^\infty \underline{T}_0^1 \mathcal{N}.$$

Now, let  $R$  denote the spacetime Riemann curvature tensor associated with  $g$ . We define the following curvature coefficients, which comprise the standard null decomposition of  $R$  with respect to our given geodesic foliation.

$$\begin{aligned} \alpha &\in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}, & \alpha(X, Y) &= R(L, X, L, Y), \\ \beta &\in \mathcal{C}^\infty \underline{T}_1^0 \mathcal{N}, & \beta(X) &= \frac{1}{2} R(L, X, L, \underline{L}), \\ \rho &\in \mathcal{C}^\infty \mathcal{N}, & \rho &= \frac{1}{4} R(L, \underline{L}, L, \underline{L}), \\ \sigma &\in \mathcal{C}^\infty \mathcal{N}, & \rho &= \frac{1}{4} {}^* R(L, \underline{L}, L, \underline{L}), \\ \underline{\beta} &\in \mathcal{C}^\infty \underline{T}_1^0 \mathcal{N}, & \underline{\beta}(X) &= \frac{1}{2} R(\underline{L}, X, \underline{L}, L), \\ \underline{\alpha} &\in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}, & \underline{\alpha}(X, Y) &= R(\underline{L}, X, \underline{L}, Y), \end{aligned}$$

where  ${}^* R$  denotes the left (spacetime) Hodge dual of  $R$ . In the vacuum setting, these coefficients comprise all the independent components of  $R$ .

By a direct computation, cf. [11], one can see that

$$(2.7) \quad k = k^{(\gamma)} = \chi.$$

We can then define the evolutionary covariant derivative  $\nabla_t$  as before. One can show that this operator  $\nabla_t$  coincides with the appropriate projection  $\nabla_L$  onto the  $\mathcal{S}_v$ 's of the spacetime covariant derivative operator  $D_L$ .<sup>25</sup>

The Ricci and curvature coefficients defined above are related to each other via a family of geometric differential equations, known as the *null structure equations*. Below, we state a few of these equations. For derivations, see, e.g., [6, 10].

<sup>25</sup>In previous works, e.g., [11, 13, 17, 18, 22, 23], the ‘‘horizontal covariant evolution’’ operation was defined by the above description. Our definition of  $\nabla_t$  here generalizes this.



**Proposition 2.5.** *The following “structure equations” hold on  $\mathcal{N}$ .*

- *Evolution equations:*

$$(2.8) \quad \begin{aligned} \nabla_t \chi_{ab} &= -\gamma^{cd} \chi_{ac} \chi_{bd} - \alpha_{ab}, \\ \nabla_t \zeta_a &= -2\gamma^{bc} \chi_{ab} \zeta_c - \beta_a. \end{aligned}$$

- *Null Bianchi equations:*

$$(2.9) \quad \nabla_t \beta_a = \gamma^{bc} \nabla_b \alpha_{ac} - 2(\gamma^{bc} \chi_{bc}) \beta_a + \gamma^{bc} \zeta_b \alpha_{ac}.$$

- *Elliptic equations:*

$$(2.10) \quad \nabla_b \chi_{ac} - \nabla_c \chi_{ab} = -\epsilon_{bc} \epsilon_a^d \beta_d + \chi_{ab} \zeta_c - \chi_{ac} \zeta_b.$$

For the full list of structure equations on geodesically foliated null cones in Einstein-vacuum spacetimes, see [11, 22]. For other foliations, see [17, 18, 25]. These equations play a fundamental role in controlling the geometry of  $\mathcal{N}$ .

In particular, note that the curl  $\mathfrak{C}$  of  $k$  is described precisely by the equation (2.10). In fact, this explicit formula for  $\mathfrak{C}$  is essential for explaining how the upcoming estimates in this paper apply to this setting of regular null cones, in the specific case of finite curvature flux bounds.

### 3. ANALYSIS ON EUCLIDEAN SPACES

Before venturing into the intended setting of this paper – surfaces with evolving geometries – we first examine the Euclidean analogues of our main results. In Section 5, we will apply these Euclidean estimates to derive our main results.

In terms of the formalisms of Section 2, here we shall examine the special case

$$\mathcal{S} = \mathbb{R}^2, \quad \mathcal{N} = [0, \delta) \times \mathbb{R}^2.$$

For the horizontal metric, we let  $\gamma \in \mathcal{C}^\infty \underline{T}_2^0 \mathcal{N}$  denote the equivariant horizontal field such that each  $\gamma[v]$  is the Euclidean metric on  $\mathbb{R}^2$ .

**Remark.** *Note that for any  $\Psi \in \mathcal{C}^\infty \underline{T}_1^r \mathcal{N}$ , both the Lie derivative  $\mathfrak{L}_t \Psi$  and the covariant derivative  $\nabla_t \Psi$  coincide with the  $t$ -partial derivative  $\partial_t \Psi$  of  $\Psi$ .*

Throughout, we will adopt the notations of the previous section whenever convenient. In particular, this applies to the integral norms we use here. Furthermore, since the only metric under consideration here is the standard Euclidean metric, we will omit entirely the symbols  $\gamma$  and  $\epsilon$  from our notations.

In this section, we will be dealing mostly with two families of functions:

- Let  $\mathcal{S}_x \mathbb{R}^2$  denote the space of “Schwartz” functions, i.e., the space of smooth functions  $f$  on  $\mathbb{R}^2$  such that  $\nabla^k f$  is rapidly decreasing for every  $k \geq 0$ .
- Also, let  $C_t^\infty \mathcal{S}_x \mathcal{N}$  denote the space of smooth functions on  $\mathcal{N}$  such that  $\nabla_t^m \phi[v] \in \mathcal{S}_x \mathbb{R}^2$  for every  $k, m \geq 0$  and  $v \in [0, \delta)$ .

Finally, throughout this section, we will assume arbitrary functions

$$\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}, \quad f \in \mathcal{S}_x \mathbb{R}^2.$$

**3.1. Littlewood-Paley Theory.** We very briefly review some basic elements of the classical dyadic Littlewood-Paley (abbreviated L-P) theory on  $\mathbb{R}^2$ . First, the standard L-P operators can be constructed in the following fashion:

- Fix a cutoff function  $\varsigma \in \mathcal{S}_x \mathbb{R}^2$  such that

$$\text{supp } \varsigma \subseteq \left\{ \xi \in \mathbb{R}^2 \mid \frac{1}{2} \leq |\xi| \leq 2 \right\},$$

and define its rescalings

$$\varsigma_k \in \mathcal{S}_x \mathbb{R}^2, \quad \varsigma_k(\xi) = \varsigma(2^{-k}\xi), \quad k \in \mathbb{Z}.$$

Furthermore, this  $\varsigma$  can be chosen such that

$$\sum_{k \in \mathbb{Z}} \varsigma_k = \chi_{\mathbb{R} \setminus \{0\}}.$$

- We define our L-P operators to be the Fourier multipliers associated with the above cutoff functions; for each  $k \in \mathbb{Z}$ , we define

$$\mathcal{F}(P_k f) = \varsigma_k \mathcal{F} f,$$

where  $\mathcal{F}$  is the usual Fourier transform on  $\mathbb{R}^2$ .

- In addition, for any  $k \in \mathbb{Z}$ , we can define the operators

$$P_{<k} = \sum_{l=-\infty}^{k-1} P_l, \quad P_{\geq k} = \sum_{l=k}^{\infty} P_l.$$

The properties satisfied by these multipliers are well-known; see, e.g., [9, 21]. Here, we list some of the basic estimates we will find useful later.

- *Almost Orthogonality:* If  $k_1, k_2 \in \mathbb{Z}$  and  $|k_1 - k_2| > 1$ , then

$$P_{k_1} P_{k_2} \equiv 0.$$

As a corollary of this, we have the comparison

$$(3.1) \quad \|f\|_{L_x^2}^2 \simeq \sum_{k \geq 0} \|P_k f\|_{L_x^2}^2 + \|P_{<0} f\|_{L_x^2}^2.$$

- *Boundedness:* For any  $q \in [1, \infty]$  and  $k \in \mathbb{Z}$ ,

$$(3.2) \quad \|P_k f\|_{L_x^q} + \|P_{<k} f\|_{L_x^q} + \|P_{\geq k} f\|_{L_x^q} \lesssim \|f\|_{L_x^q}.$$

- *Finite Band:* For any  $q \in [1, \infty]$  and  $k \in \mathbb{Z}$ ,

$$(3.3) \quad \begin{aligned} \|\nabla P_{<0} f\|_{L_x^q} + 2^{-k} \|\nabla P_k f\|_{L_x^q} &\lesssim \|f\|_{L_x^q}, \\ 2^k \|P_k f\|_{L_x^q} &\lesssim \|\nabla f\|_{L_x^q}. \end{aligned}$$

- *Bernstein Inequalities:* If  $q, q' \in [1, \infty]$ ,  $q \leq q'$ , and  $k \in \mathbb{Z}$ , then

$$(3.4) \quad \|P_{<0} f\|_{L_x^{q'}} \lesssim \|f\|_{L_x^q}, \quad \|P_k f\|_{L_x^{q'}} \lesssim 2^{2k(\frac{1}{q} - \frac{1}{q'})} \|f\|_{L_x^q}.$$

**Remark.** As is usual, we define the operators  $P_k, P_{\geq k}, P_{<k}$  on tensorial quantities by defining them componentwise, with respect to the standard Euclidean coordinates. Then, the above properties (3.1)-(3.4) extend directly to tensor fields. Note also that the L-P operators commute with both  $\nabla$  and  $\nabla_t$ .

We also highlight some common tricks involving L-P operators:

- From the almost orthogonality property of the  $P_k$ 's, we can write

$$P_k = P_k P_{k-1} + P_k P_k + P_k P_{k+1}, \quad k \in \mathbb{Z}.$$

As a result of the above, we will often use the schematic notation

$$P_{\sim k} = \sum_{l=k-c}^{k+c} P_l$$

for some universal small positive integer  $c$ . For instance, we can write

$$P_k = P_k P_{\sim k}.$$

In practice, this manipulation is extremely useful when one wants to apply the estimates (3.2)-(3.4) without destroying the  $P_k$ 's.

- We sometimes also employ the summed schematic quantities

$$P_{\lesssim k} = \sum_{l=-\infty}^{k-1} P_{\sim l}, \quad P_{\gtrsim k} = \sum_{l=k}^{\infty} P_{\sim l}, \quad k \in \mathbb{Z}.$$

- Another important point is to observe that we can write

$$2^k P_k \sim \nabla \tilde{P}_k, \quad k \in \mathbb{Z},$$

where the operator  $\tilde{P}_k$  is like  $P_k$  but is constructed from a slightly different Fourier multiplier. This new operator  $\tilde{P}_k$  satisfies the many of the same properties as  $P_k$ , including the basic estimates (3.2)-(3.4), although with different constants. Furthermore, since  $\tilde{P}_k$  has the same Fourier support as  $P_k$ , then we can combine this with the previous point to obtain

$$\tilde{P}_k = \tilde{P}_k P_{\sim k}, \quad k \in \mathbb{Z}.$$

We will often employ these tricks without further mention.

We can easily extend these L-P operators to act on functions on  $\mathcal{N}$ :

$$(P_k \phi)[v] = P_k(\phi[v]), \quad k \in \mathbb{Z},$$

and similarly for  $P_{<k}$  and  $P_{\geq k}$ . In other words, we take the appropriate Fourier localization of  $\phi$  with respect to only the spatial components.

L-P operators provide a practical way to express Besov norms on  $\mathbb{R}^2$ . For instance, for any  $s \in [0, \infty)$  and  $p \in [1, \infty]$ , we can define the following  $B_{2,1}^s$ -norms:

$$\begin{aligned} \|f\|_{B_x^s} &= \sum_{k \geq 0} 2^{sk} \|P_k f\|_{L_x^2} + \|P_{<0} f\|_{L_x^2} \\ \|\phi\|_{B_{t,x}^{p,s}} &= \sum_{k \geq 0} 2^{sk} \|P_k \phi\|_{L_{t,x}^{p,2}} + \|P_{<0} \phi\|_{L_{t,x}^{p,2}}. \end{aligned}$$

In other words, one replaces the usual  $\ell^2$ -summation of L-P projections in Sobolev norms by an  $\ell^1$ -summation. While the Sobolev space  $H^1(\mathbb{R}^2)$  barely fails to embed into  $L^\infty(\mathbb{R}^2)$ , the above Besov-type spaces address this deficiency:

$$\begin{aligned} (3.5) \quad \|f\|_{L_x^\infty} &\lesssim \|f\|_{B_x^1} \lesssim \|\nabla f\|_{B_x^0} + \|f\|_{L_x^2}, \\ \|\phi\|_{L_{t,x}^{\infty,\infty}} &\lesssim \|\phi\|_{B_{t,x}^{\infty,1}} \lesssim \|\nabla \phi\|_{B_{t,x}^{\infty,0}} + \|\phi\|_{L_{t,x}^{\infty,2}}. \end{aligned}$$

This can be extremely useful, or even essential, in low-regularity settings.

**3.2. Calculus Estimates.** Here, we prove some basic calculus estimates that will be quite useful in our upcoming analysis. First, we discuss some basic Gagliardo-Nirenberg-type estimate for functions on  $\mathcal{N}$ .

**Proposition 3.1.** *The following estimates hold for any  $\phi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ :*

$$(3.6) \quad \delta^{\frac{1}{2}} \|\phi\|_{L_{x,t}^{2,\infty}} \lesssim (\delta \|\nabla_t \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{t,x}^{2,2}})^{\frac{1}{2}} \|\phi\|_{L_{t,x}^{2,2}}^{\frac{1}{2}},$$

$$(3.7) \quad \delta^{\frac{1}{2}} \|\phi\|_{L_{x,t}^{4,\infty}} \lesssim (\delta \|\nabla_t \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{t,x}^{2,2}})^{\frac{1}{2}} \|\nabla \phi\|_{L_{t,x}^{2,2}}^{\frac{1}{2}}.$$

*Proof.* Fix two cutoff functions  $\eta_+, \eta_-$  defined on  $(0, \delta)$  satisfying <sup>26</sup>

$$\eta_+(v) = \begin{cases} 1 & \frac{\delta}{2} \leq v < \delta, \\ 0 & 0 < v \leq \frac{\delta}{4}, \end{cases} \quad \eta_- = 1 - \eta_+.$$

Fix a point  $x \in \mathbb{R}^2$ . If  $\delta/2 \leq v < \delta$ , then we have the estimate

$$\begin{aligned} |\phi(v, x)|^2 &= \int_0^v \frac{\partial}{\partial w} [\eta_+(w) |\phi(w, x)|^2] dw \\ &\lesssim \int_0^\delta [|\nabla_t \phi(w, x)| |\phi(w, x)| + \delta^{-1} |\phi(w, x)|^2] dw. \end{aligned}$$

Similarly, if  $0 < v \leq \delta/2$ , then

$$\begin{aligned} |\phi(v, x)|^2 &= - \int_v^\delta \frac{\partial}{\partial w} [\eta_-(w) |\phi(w, x)|^2] dw \\ &\lesssim \int_0^\delta [|\nabla_t \phi(w, x)| |\phi(w, x)| + \delta^{-1} |\phi(w, x)|^2] dw. \end{aligned}$$

Taking the supremum over all  $v \in [0, \delta]$  yields

$$\sup_{0 \leq v < \delta} |\phi(v, x)|^2 \lesssim \int_0^\delta [|\nabla_t \phi(v, x)| |\phi(v, x)| + \delta^{-1} |\phi(v, x)|^2] dv.$$

Integrating the above over  $\mathbb{R}^2$  results in (3.6).

Next, for (3.7), we again use  $\eta_+$  and  $\eta_-$  as before to derive

$$\sup_{0 \leq v < \delta} |\phi(v, x)|^4 \lesssim \int_0^\delta [|\nabla_t \phi(v, x)| |\phi(v, x)|^3 + \delta^{-1} |\phi(v, x)|^4] dv.$$

Integrating the above over  $\mathbb{R}^2$  and applying Hölder's inequality yields

$$\|\phi\|_{L_{x,t}^{4,\infty}} \lesssim (\|\nabla_t \phi\|_{L_{t,x}^{2,2}} + \delta^{-1} \|\phi\|_{L_{t,x}^{2,2}})^{\frac{1}{4}} \|\phi\|_{L_{t,x}^{6,6}}^{\frac{3}{4}}.$$

The standard Gagliardo-Nirenberg-Sobolev inequality yields for any  $v$  that

$$\|\phi[v]\|_{L_x^6}^6 \lesssim \|\nabla(|\phi|^3)[v]\|_{L_x^1}^2 \lesssim \|\nabla \phi[v]\|_{L_x^2}^2 \|\phi[v]\|_{L_x^4}^4.$$

Integrating the above over the time interval  $[0, \delta]$ , we obtain

$$\begin{aligned} \|\phi\|_{L_{x,t}^{4,\infty}} &\lesssim (\|\nabla_t \phi\|_{L_{t,x}^{2,2}} + \delta^{-1} \|\phi\|_{L_{t,x}^{2,2}})^{\frac{1}{4}} \|\phi\|_{L_{t,x}^{6,6}}^{\frac{3}{4}} \\ &\lesssim (\|\nabla_t \phi\|_{L_{t,x}^{2,2}} + \delta^{-1} \|\phi\|_{L_{t,x}^{2,2}})^{\frac{1}{4}} \|\nabla \phi\|_{L_{t,x}^{2,2}}^{\frac{1}{4}} \|\phi\|_{L_{x,t}^{4,\infty}}^{\frac{1}{2}}, \end{aligned}$$

and the desired estimate (3.7) follows.  $\square$

<sup>26</sup>We can define  $\eta_\pm$  to be suitable rescalings of cutoff functions for the case  $\delta = 1$ .

In anticipation of future L-P estimates involving the  $H_{t,x}^1$ -norm, we define the following schematic quantities, which are analogous to the “envelopes” defined in [13, 22]. Given a nonnegative integer  $k$ , we define

$$\begin{aligned} N_k \phi &= 2^k \|P_{\sim k} \phi\|_{L_{t,x}^{2,2}} + \delta \|P_{\sim k} \nabla_t \phi\|_{L_{t,x}^{2,2}}, \\ E_k \phi &= \sum_{l \geq 0} 2^{-\frac{1}{2}|k-l|} N_l \phi + 2^{-\frac{k}{2}} \|\phi\|_{H_{t,x}^1}. \end{aligned}$$

Notice that  $N_k \phi \leq E_k \phi$  for every  $k \geq 0$ . In upcoming estimates, we will exploit these quantities in order to dyadically recover the  $H_{t,x}^1$ -norm.

**Proposition 3.2.** *The following estimates hold for any  $\phi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ :*

$$(3.8) \quad \sum_{k \geq 0} (N_k \phi)^2 \lesssim \|\phi\|_{H_{t,x}^1}^2, \quad \sum_{k \geq 0} (E_k \phi)^2 \lesssim \|\phi\|_{H_{t,x}^1}^2.$$

*Proof.* The first estimate is a direct consequence of (3.1). For the second inequality, we apply Cauchy’s inequality in order to obtain

$$\begin{aligned} \sum_{k \geq 0} (E_k \phi)^2 &\lesssim \sum_{k \geq 0} \sum_{l \geq 0} 2^{-\frac{1}{2}|k-l|} N_l \phi \sum_{l' \geq 0} 2^{-\frac{1}{2}|k-l'|} N_{l'} \phi + \sum_{k \geq 0} 2^{-k} \|\phi\|_{H_{t,x}^1}^2 \\ &\lesssim \sum_{k \geq 0} \sum_{l \geq 0} 2^{-\frac{1}{2}|k-l|} \sum_{l' \geq 0} 2^{-\frac{1}{2}|k-l'|} [(N_l \phi)^2 + (N_{l'} \phi)^2] + \|\phi\|_{H_{t,x}^1}^2 \\ &\lesssim \sum_{k \geq 0} \sum_{l \geq 0} 2^{-\frac{1}{2}|k-l|} (N_l \phi)^2 \sum_{l' \geq 0} 2^{-\frac{1}{2}|k-l'|} + \|\phi\|_{H_{t,x}^1}^2. \end{aligned}$$

Summing the first term on the right-hand side, first over  $l'$  and then  $k$ , we have

$$\sum_{k \geq 0} (E_k \phi)^2 \lesssim \sum_{l \geq 0} (N_l \phi)^2 + \|\phi\|_{H_{t,x}^1}^2.$$

Applying the first estimate in (3.8) completes the proof.  $\square$

We can apply Proposition 3.2 to prove dyadic modifications of Proposition 3.1.

**Proposition 3.3.** *For any  $k \geq 0$  and  $\phi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ ,*

$$(3.9) \quad \delta^{\frac{1}{2} - \frac{1}{p}} \|P_{\sim k} \phi\|_{L_{t,x}^{p,q}} \lesssim 2^{(\frac{1}{2} - \frac{2}{q} - \frac{1}{p})k} N_k \phi, \quad \delta^{\frac{1}{2} - \frac{1}{p}} \|P_{\leq 0} \phi\|_{L_{t,x}^{p,q}} \lesssim \|\phi\|_{H_{t,x}^1}.$$

*Proof.* Consider first the case  $p = 2$ . By (3.4), we obtain as desired

$$\|P_{\sim k} \phi\|_{L_{t,x}^{2,q}} \lesssim 2^{k(1 - \frac{2}{q})} \|P_{\sim k} \phi\|_{L_{t,x}^{2,2}} = 2^{-\frac{2k}{q}} N_k \phi.$$

The corresponding estimate for  $P_{\leq 0} \phi$  is similar.

Consider next the case  $p = \infty$ . Applying (3.6) along with (3.4), we have

$$\begin{aligned} \delta^{\frac{1}{2}} \|P_{\sim k} \phi\|_{L_{t,x}^{\infty,q}} &\lesssim 2^{k(1 - \frac{2}{q})} (\delta \|P_{\sim k} \nabla_t \phi\|_{L_{t,x}^{2,2}} + \|P_{\sim k} \phi\|_{L_{t,x}^{2,2}})^{\frac{1}{2}} \|P_{\sim k} \phi\|_{L_{t,x}^{2,2}}^{\frac{1}{2}} \\ &\lesssim 2^{(\frac{1}{2} - \frac{2}{q})k} N_k \phi. \end{aligned}$$

A similar argument proves the corresponding bound for  $P_{\leq 0} \phi$ . Now, for general  $p$ , one simply interpolates between the cases  $p = 2$  and  $p = \infty$ .  $\square$

**3.3. Non-Integrated Product Estimates.** We now begin the process of establishing the Euclidean analogues of our main bilinear product estimates. We begin here with the “simple” non-integrated bilinear product estimates, for which we will require an L-P decomposition of only one of the factors.

The key technical point here is a set of “intertwining” estimates, given in the subsequent lemma. These estimates involve a product operation sandwiched between two L-P operators. The idea is that one can treat them differently depending on which of the L-P operators involves a higher frequency.

**Lemma 3.4.** *Fix  $k, l \geq 0$ ; let  $f, g \in \mathcal{S}_x \mathbb{R}^2$ ; and let  $\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ .*

- *The following estimates hold:*

$$(3.10) \quad \begin{aligned} \|P_k(f \cdot P_l g)\|_{L_x^2} &\lesssim 2^{-|k-l|}(\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty})\|P_{\sim l} g\|_{L_x^2}, \\ \|P_k(f \cdot P_{<0} g)\|_{L_x^2} &\lesssim 2^{-k}(\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty})\|g\|_{L_x^2}. \end{aligned}$$

- *If  $\psi$  is  $t$ -parallel, then <sup>27</sup>*

$$(3.11) \quad \begin{aligned} \|P_k(\phi \cdot P_l \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-|k-l|}(\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}})\|P_{\sim l} \psi[0]\|_{L_x^2}, \\ \|P_k(\phi \cdot P_{<0} \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-k}(\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}})\|\psi[0]\|_{L_x^2}. \end{aligned}$$

*Proof.* Suppose first that  $l \geq k$ . The main step here is to “create” a derivative from the higher frequency projection  $P_l$ . In the case of (3.10), we obtain

$$\begin{aligned} \|P_k(f \cdot P_l g)\|_{L_x^2} &\lesssim 2^{-l}\|P_k(f \cdot \nabla \tilde{P}_l g)\|_{L_x^2} \\ &\lesssim 2^{-l}[\|P_k \nabla(f \cdot \tilde{P}_l g)\|_{L_x^2} + \|P_k(\nabla f \cdot \tilde{P}_l g)\|_{L_x^2}]. \end{aligned}$$

The two terms on the right-hand side are treated using (3.3) and (3.4):

$$\begin{aligned} \|P_k(f \cdot P_l g)\|_{L_x^2} &\lesssim 2^{k-l}[\|f \cdot \tilde{P}_l g\|_{L_x^2} + \|\nabla f \cdot \tilde{P}_l g\|_{L_x^1}] \\ &\lesssim 2^{-|k-l|}(\|f\|_{L_x^\infty} + \|\nabla f\|_{L_x^2})\|P_{\sim l} g\|_{L_x^2}. \end{aligned}$$

Similarly, for (3.11), we have

$$\begin{aligned} \|P_k(\phi \cdot P_l \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-l}[\|P_k \nabla(\phi \cdot \tilde{P}_l \psi)\|_{L_{t,x}^{2,2}} + \|P_k(\nabla \phi \cdot \tilde{P}_l \psi)\|_{L_{t,x}^{2,2}}] \\ &\lesssim 2^{k-l}[\|\phi \cdot \tilde{P}_l \psi\|_{L_{t,x}^{2,2}} + \|\nabla \phi \cdot \tilde{P}_l \psi\|_{L_{t,x}^{2,1}}] \\ &\lesssim 2^{-|k-l|}(\|\phi\|_{L_{x,t}^{\infty,2}} + \|\nabla \phi\|_{L_{t,x}^{2,2}})\|P_{\sim l} \psi\|_{L_{x,t}^{2,\infty}}, \end{aligned}$$

where we once again applied (3.3) and (3.4). This estimate is completed by the observation that since  $P_{\sim l} \psi$  is independent of the  $t$ -variable, then

$$\|P_{\sim l} \psi\|_{L_{x,t}^{2,\infty}} = \|P_{\sim l} \psi[0]\|_{L_x^2}.$$

Next, we consider the other case  $l < k$ . First, for (3.10), we apply (3.3):

$$\begin{aligned} \|P_k(f \cdot P_l g)\|_{L_x^2} &\lesssim 2^{-k}[\|\nabla(f \cdot P_l g)\|_{L_x^2} \\ &\lesssim 2^{-k}(\|\nabla f\|_{L_x^2}\|P_l g\|_{L_x^\infty} + \|f\|_{L_x^\infty}\|\nabla P_l g\|_{L_x^2}). \end{aligned}$$

By (3.3) and (3.4), then

$$\begin{aligned} \|P_k(f \cdot P_l g)\|_{L_x^2} &\lesssim 2^{-k+l}(\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty})\|P_{\sim l} g\|_{L_x^2} \\ &= 2^{-|k-l|}(\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty})\|P_{\sim l} g\|_{L_x^2}. \end{aligned}$$

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<sup>27</sup>In other words, if  $\psi(v, x) = \psi(0, x)$  for every  $v \in [0, \delta)$  and  $x \in \mathbb{R}^2$ .

Similarly, for (3.11), we apply (3.3) and (3.4) once again to obtain

$$\begin{aligned} \|P_k(\phi \cdot P_l \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-k}(\|\nabla \phi\|_{L_{t,x}^{2,2}}\|P_l \psi\|_{L_{t,x}^{\infty,\infty}} + \|\phi\|_{L_{x,t}^{\infty,2}}\|\nabla P_l \psi\|_{L_{x,t}^{2,\infty}}) \\ &\lesssim 2^{-|k-l|}(\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}})\|P_{\sim l} \psi\|_{L_x^2}. \end{aligned}$$

Once again, we used that  $\psi$  and  $\nabla P_{\sim l} \psi$  are independent of  $t$ .

Finally, the low-frequency versions of (3.10) and (3.11) involving  $P_{<0}$  can be similarly proved by analogous applications of (3.3) and (3.4).  $\square$

We now apply Lemma 3.4 to prove the desired product estimates.

**Theorem 3.5.** *Let  $f, g \in \mathcal{S}_x \mathbb{R}^2$ , and let  $\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ .*

- *If  $p \in [1, \infty]$ , then*

$$(3.12) \quad \begin{aligned} \|f \cdot g\|_{B_x^0} &\lesssim (\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty})\|g\|_{B_x^0}, \\ \|\phi \cdot \psi\|_{B_{t,x}^{p,0}} &\lesssim (\|\nabla \phi\|_{L_{t,x}^{\infty,2}} + \|\phi\|_{L_{t,x}^{\infty,\infty}})\|\psi\|_{B_{t,x}^{p,0}}. \end{aligned}$$

- *If  $\psi$  is  $t$ -parallel, then*

$$(3.13) \quad \|\phi \cdot \psi\|_{B_{t,x}^{2,0}} \lesssim (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}})\|\psi[0]\|_{B_x^0}.$$

*Proof.* For the first part of (3.12), we begin with the decomposition

$$\|fg\|_{B_x^0} \lesssim \sum_{k,l \geq 0} \|P_k(fP_l g)\|_{L_x^2} + \sum_{k \geq 0} \|P_k(fP_{<0} g)\|_{L_x^2} + \|P_{<0}(fg)\|_{L_x^2}.$$

The last term on the right-hand side is bounded trivially using Hölder's inequality:

$$\|P_{<0}(fg)\|_{L_x^2} \lesssim \|f\|_{L_x^\infty} \|g\|_{L_x^2}.$$

For the remaining terms, we apply (3.10) and sum over  $k$ :

$$\begin{aligned} \sum_{k,l \geq 0} \|P_k(fP_l g)\|_{L_x^2} &\lesssim (\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty}) \sum_{k,l \geq 0} 2^{-|k-l|} \|P_{\sim l} g\|_{L_x^2} \\ &\lesssim (\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty}) \sum_{l \geq 0} \|P_{\sim l} g\|_{L_x^2}, \\ \sum_{k \geq 0} \|P_k(fP_{<0} g)\|_{L_x^2} &\lesssim (\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty}) \sum_{k \geq 0} 2^{-k} \|g\|_{L_x^2} \\ &\lesssim (\|\nabla f\|_{L_x^2} + \|f\|_{L_x^\infty}) \|g\|_{L_x^2}. \end{aligned}$$

The first estimate in (3.12) follows immediately.

One can, via a completely analogous process, derive the remaining estimate of (3.12).<sup>28</sup> Finally, the estimate (3.13) is again similarly proved, using (3.11).  $\square$

**3.4. Integrated Product Estimates.** Next, we look at simple *integrated* bilinear product estimates, which are similar to those in Theorem 3.5, but also contain the  $t$ -integral operator  $\mathfrak{T}$ . The key steps once again involve intertwining estimates.

**Remark.** *These integrated estimates, in the Euclidean case, can also be found in [13, Sect. 3]. We include them here for convenience and completeness.*

**Lemma 3.6.** *Fix  $k, l \geq 0$ , and let  $\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ .*

<sup>28</sup>To obtain the necessary intertwining estimates in this case, one first applies (3.10) to each  $\phi[v]$  and  $\psi[v]$  and then takes the  $L^p$ -norms of these estimates with respect to  $t$ .

• The following estimates hold:

$$(3.14) \quad \begin{aligned} \|P_k \mathfrak{T}(\phi \cdot P_l \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-|k-l|} (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|P_{\sim l} \psi\|_{L_{t,x}^{2,2}}, \\ \|P_k \mathfrak{T}(\phi \cdot P_{<0} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-k} (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|\psi\|_{L_{t,x}^{2,2}}. \end{aligned}$$

• The following estimates hold:

$$(3.15) \quad \begin{aligned} \|P_k(\phi \cdot \mathfrak{T} P_l \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-|k-l|} (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|P_{\sim l} \psi\|_{L_{t,x}^{1,2}}, \\ \|P_k(\phi \cdot \mathfrak{T} P_{<0} \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-k} (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|\psi\|_{L_{t,x}^{1,2}}. \end{aligned}$$

*Proof.* First, assume  $l \geq k$ . For (3.14), we have

$$\begin{aligned} \|P_k \mathfrak{T}(\phi \cdot P_l \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-l} [\|P_k \nabla \mathfrak{T}(\phi \cdot \tilde{P}_l \psi)\|_{L_{t,x}^{\infty,2}} + \|P_k \mathfrak{T}_v(\nabla \phi \cdot \tilde{P}_l \psi)\|_{L_{t,x}^{\infty,2}}] \\ &\lesssim 2^{k-l} [\|\mathfrak{T}(\phi \cdot \tilde{P}_l \psi)\|_{L_{x,t}^{2,\infty}} + \|\mathfrak{T}(\nabla \phi \cdot \tilde{P}_l \psi)\|_{L_{x,t}^{1,\infty}}], \end{aligned}$$

where in the last step, we applied (3.3) and (3.4). By the definition of  $\mathfrak{T}$ , then

$$\begin{aligned} \|P_k \mathfrak{T}(\phi \cdot P_l \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{k-l} [\|\phi \cdot \tilde{P}_l \psi\|_{L_{x,t}^{2,1}} + \|\nabla \phi \cdot \tilde{P}_l \psi\|_{L_{x,t}^{1,1}}] \\ &\lesssim 2^{-|k-l|} (\|\phi\|_{L_{x,t}^{\infty,2}} + \|\nabla \phi\|_{L_{t,x}^{2,2}}) \|P_{\sim l} \psi\|_{L_{t,x}^{2,2}}, \end{aligned}$$

as desired. Similarly, in the case of (3.15), we have

$$\begin{aligned} \|P_k(\phi \cdot \mathfrak{T} P_l \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-l} [\|P_k \nabla(\phi \cdot \mathfrak{T} \tilde{P}_l \psi)\|_{L_{t,x}^{2,2}} + \|P_k(\nabla \phi \cdot \mathfrak{T} \tilde{P}_l \psi)\|_{L_{t,x}^{2,2}}] \\ &\lesssim 2^{k-l} (\|\phi \cdot \mathfrak{T} \tilde{P}_l \psi\|_{L_{t,x}^{2,2}} + \|\nabla \phi \cdot \mathfrak{T} \tilde{P}_l \psi\|_{L_{t,x}^{2,1}}) \\ &\lesssim 2^{k-l} (\|\phi\|_{L_{x,t}^{\infty,2}} + \|\nabla \phi\|_{L_{t,x}^{2,2}}) \|\mathfrak{T} \tilde{P}_l \psi\|_{L_{x,t}^{2,\infty}}. \end{aligned}$$

By the definition of  $\mathfrak{T}$  and the Minkowski integral inequality, we obtain

$$\|P_k(\phi \cdot \mathfrak{T} P_l \psi)\|_{L_{t,x}^{2,2}} \lesssim 2^{-|k-l|} (\|\phi\|_{L_{x,t}^{\infty,2}} + \|\nabla \phi\|_{L_{t,x}^{2,2}}) \|P_{\sim l} \psi\|_{L_{t,x}^{1,2}}.$$

Next, assume  $l < k$ . For (3.14), first

$$\begin{aligned} \|P_k \mathfrak{T}(\phi \cdot P_l \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-k} [\|\mathfrak{T}(\nabla \phi \cdot P_l \psi)\|_{L_{t,x}^{\infty,2}} + \|\mathfrak{T}(\phi \cdot \nabla P_l \psi)\|_{L_{t,x}^{\infty,2}}] \\ &\lesssim 2^{-k} (\|\nabla \phi \cdot P_l \psi\|_{L_{x,t}^{2,1}} + \|\phi \cdot \nabla P_l \psi\|_{L_{x,t}^{2,1}}). \end{aligned}$$

Applying (3.3) and (3.4) yet again yields

$$\begin{aligned} \|P_k \mathfrak{T}(\phi \cdot P_l \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-k} (\|\nabla \phi\|_{L_{t,x}^{2,2}} \|P_l \psi\|_{L_{t,x}^{2,\infty}} + \|\phi\|_{L_{x,t}^{\infty,2}} \|\nabla P_l \psi\|_{L_{t,x}^{2,2}}) \\ &\lesssim 2^{-|k-l|} (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|P_{\sim l} \psi\|_{L_{t,x}^{2,2}}. \end{aligned}$$

Similarly, for (3.15), we have

$$\begin{aligned} \|P_k(\phi \cdot \mathfrak{T} P_l \psi)\|_{L_{t,x}^{2,2}} &\lesssim 2^{-k} [\|\nabla \phi \cdot \mathfrak{T} P_l \psi\|_{L_{t,x}^{2,2}} + \|\phi \cdot \mathfrak{T} \nabla P_l \psi\|_{L_{t,x}^{2,2}}] \\ &\lesssim 2^{-k} (\|\nabla \phi\|_{L_{t,x}^{2,2}} \|\mathfrak{T} P_l \psi\|_{L_{x,t}^{\infty,\infty}} + \|\phi\|_{L_{x,t}^{\infty,2}} \|\mathfrak{T} \nabla P_l \psi\|_{L_{x,t}^{2,\infty}}) \\ &\lesssim 2^{-k} (\|\nabla \phi\|_{L_{t,x}^{2,2}} \|P_l \psi\|_{L_{x,t}^{\infty,1}} + \|\phi\|_{L_{x,t}^{\infty,2}} \|\nabla P_l \psi\|_{L_{x,t}^{2,1}}) \\ &\lesssim 2^{-|k-l|} (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|P_{\sim l} \psi\|_{L_{t,x}^{1,2}}. \end{aligned}$$

Finally, the versions of (3.14) and (3.15) involving  $P_{<0}$  are similarly proved.  $\square$



**Theorem 3.7.** *For any  $\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ ,*

$$(3.16) \quad \|\mathfrak{T}(\phi \cdot \psi)\|_{B_{t,x}^{\infty,0}} \lesssim (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|\psi\|_{B_{t,x}^{2,0}},$$

$$(3.17) \quad \|\phi \cdot \mathfrak{T}\psi\|_{B_{t,x}^{2,0}} \lesssim (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|\psi\|_{B_{t,x}^{1,0}}.$$

*Proof.* For (3.16), we begin with the decomposition

$$\begin{aligned} \|\mathfrak{T}(\phi\psi)\|_{B_{t,x}^{\infty,0}} &\lesssim \sum_{k,l \geq 0} \|P_k \mathfrak{T}(\phi P_l \psi)\|_{L_{t,x}^{\infty,2}} + \sum_{k \geq 0} \|P_k \mathfrak{T}(\phi P_{<0} \psi)\|_{L_{t,x}^{\infty,2}} \\ &\quad + \|P_{<0} \mathfrak{T}(\phi\psi)\|_{L_{t,x}^{\infty,2}}. \end{aligned}$$

The last term on the right-hand side is trivially bounded:

$$\|P_{<0} \mathfrak{T}(\phi\psi)\|_{L_{t,x}^{\infty,2}} \lesssim \|\phi\psi\|_{L_{t,t}^{2,1}} \lesssim \|\phi\|_{L_{x,t}^{\infty,2}} \|\psi\|_{L_{t,x}^{2,2}}.$$

For the nontrivial terms, we apply the intertwining estimates (3.14):

$$\begin{aligned} \sum_{k,l \geq 0} \|P_k \mathfrak{T}(\phi P_l \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \sum_{k,l \geq 0} 2^{-|k-l|} \|P_{\sim l} \psi\|_{L_{t,x}^{2,2}} \\ &\lesssim (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \sum_{l \geq 0} \|P_{\sim l} \psi\|_{L_{t,x}^{2,2}}, \\ \sum_{k \geq 0} \|P_k \mathfrak{T}(\phi P_{<0} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \sum_{k \geq 0} 2^{-k} \|\psi\|_{L_{t,x}^{2,2}} \\ &\lesssim (\|\nabla \phi\|_{L_{t,x}^{2,2}} + \|\phi\|_{L_{x,t}^{\infty,2}}) \|\psi\|_{L_{t,x}^{2,2}}. \end{aligned}$$

The above inequalities imply (3.16). The other estimate (3.17) is established in an analogous fashion, using the intertwining estimate (3.15).  $\square$

**3.5. Envelope Estimates.** Next, we consider analogues to Lemmas 3.4 and 3.6, but which require L-P decompositions for both factors. While the right-hand sides of the intertwining estimates in Lemmas 3.4 and 3.6 contain Besov-type norms, in the following estimates, the right-hand sides will only contain  $L_{t,x}^{2,2}$ -type norms. This infinitesimal gain comes from the presence of  $t$ -derivatives.

Here, we include relatively brief accounts of the proofs of these estimates for the sake of completeness, as many of these can also be found in [13].

**Lemma 3.8.** *Fix integers  $k, l, m \geq 0$ .*

- *If  $\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ , then*

$$(3.18) \quad \begin{aligned} \delta \|P_k(P_{\sim l} \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-\frac{1}{2}|k-l| - \frac{1}{2}|k-m|} N_l \phi N_m \psi, \\ \delta \|P_k(P_{\sim l} \phi \cdot P_{\lesssim 0} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-\frac{1}{2}|k-l| - \frac{1}{2}k} N_l \phi \|\psi\|_{H_{t,x}^1}, \\ \delta \|P_k(P_{\lesssim 0} \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-\frac{1}{2}k - \frac{1}{2}|k-m|} \|\phi\|_{H_{t,x}^1} N_m \psi. \end{aligned}$$

- *If  $\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ , then*

$$(3.19) \quad \begin{aligned} \delta \|P_k \mathfrak{T}(P_{\sim l} \nabla_t \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-\frac{1}{2}|k-l| - \frac{1}{2}|k-m|} N_l \phi N_m \psi, \\ \delta \|P_k \mathfrak{T}(P_{\sim l} \nabla_t \phi \cdot P_{\lesssim 0} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-\frac{1}{2}|k-l| - \frac{1}{2}k} N_l \phi \|\psi\|_{H_{t,x}^1}, \\ \delta \|P_k \mathfrak{T}(P_{\lesssim 0} \nabla_t \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}} &\lesssim 2^{-\frac{1}{2}k - \frac{1}{2}|k-m|} \|\phi\|_{H_{t,x}^1} N_m \psi. \end{aligned}$$

*Proof.* Let  $A$  and  $B$  denote the following left-hand sides of (3.18) and (3.19):

$$A = \delta \|P_k(P_{\sim l}\phi \cdot P_{\sim m}\psi)\|_{L_{t,x}^{\infty,2}}, \quad B = \delta \|P_k\mathfrak{T}(P_{\sim l}\nabla_t\phi \cdot P_{\sim m}\psi)\|_{L_{t,x}^{\infty,2}}.$$

Consider first the case  $l \geq k$  and  $m \geq k$ . By (3.4),

$$A \lesssim \delta 2^k \|P_{\sim l}\phi \cdot P_{\sim m}\psi\|_{L_{t,x}^{\infty,1}} \lesssim \delta 2^k \|P_{\sim l}\phi\|_{L_{t,x}^{\infty,2}} \|P_{\sim m}\psi\|_{L_{t,x}^{\infty,2}}.$$

Applying (3.9) yields

$$A \lesssim 2^{\frac{1}{2}(k-l)} N_l \phi \cdot 2^{\frac{1}{2}(k-m)} N_m \psi \lesssim 2^{-\frac{1}{2}|k-l| - \frac{1}{2}|k-m|} N_l \phi N_m \psi.$$

For  $B$ , we split into additional cases. If  $k \leq l \leq m$ , then by (3.4) and (3.9),

$$\begin{aligned} B &\lesssim \delta 2^k \|P_{\sim l}\nabla_t\phi \cdot P_{\sim m}\psi\|_{L_{t,x}^{1,1}} \\ &\lesssim \delta 2^k \|P_{\sim l}\nabla_t\phi\|_{L_{t,x}^{2,2}} \|P_{\sim m}\psi\|_{L_{t,x}^{2,2}} \\ &\lesssim 2^{\frac{1}{2}(k-l)} N_l \phi \cdot 2^{\frac{1}{2}(k+l)-m} N_m \psi \\ &\lesssim 2^{-\frac{1}{2}|k-l| - \frac{1}{2}|k-m|} N_l \phi N_m \psi. \end{aligned}$$

On the other hand, if  $k \leq m \leq l$ , then we can reduce to the previous case by moving the “ $\nabla_t$ ” to the other factor via an integration by parts:

$$B \lesssim \delta \|P_k\mathfrak{T}(P_{\sim l}\phi \cdot P_{\sim m}\nabla_t\psi)\|_{L_{t,x}^{\infty,2}} + \delta \|P_k(P_{\sim l}\phi \cdot P_{\sim m}\psi)\|_{L_{t,x}^{\infty,2}}.$$

By symmetry, the first term on the right-hand side can be handled in the same way as the preceding case  $k \leq l \leq m$ . The second term on the right-hand side is of the same form as  $A$ , so we can handle this in the same manner as before.

Next, consider the case  $l \leq k \leq m$ . Applying Hölder’s inequality and (3.9), then

$$\begin{aligned} A &\lesssim \delta \|P_{\sim l}\phi\|_{L_{t,x}^{\infty,\infty}} \|P_{\sim m}\psi\|_{L_{t,x}^{\infty,2}} \\ &\lesssim 2^{\frac{l}{2}} N_l \phi \cdot 2^{-\frac{m}{2}} N_m \psi \\ &= 2^{-\frac{1}{2}|k-l| - \frac{1}{2}|k-m|} N_l \phi N_m \psi. \end{aligned}$$

Similarly, for  $B$ , absorbing the  $\mathfrak{T}$  into the integral norm, we have

$$B \lesssim \delta \|P_{\sim l}\nabla_t\phi\|_{L_{x,t}^{\infty,2}} \|P_{\sim m}\psi\|_{L_{t,x}^{2,2}} \lesssim 2^{-k} 2^l \delta \|P_{\sim l}\nabla_t\phi\|_{L_{t,x}^{2,2}} 2^k \|P_{\sim m}\psi\|_{L_{t,x}^{2,2}}.$$

Another application of (3.9) yields

$$B \lesssim 2^{l-k} N_l \phi \cdot 2^{k-m} N_m \psi \lesssim 2^{-|k-l| - |k-m|} N_l \phi N_m \psi.$$

For the case  $m \leq k \leq l$ , the quantity  $A$  can be controlled in the same way as in the preceding  $l \leq k \leq m$  case by symmetry, since one can trivially interchange  $l$  and  $m$ . For  $B$ , we must once again integrate by parts:

$$B \lesssim \delta \|P_k\mathfrak{T}(P_{\sim l}\phi \cdot P_{\sim m}\nabla_t\psi)\|_{L_{t,x}^{\infty,2}} + \delta \|P_k(P_{\sim l}\phi \cdot P_{\sim m}\psi)\|_{L_{t,x}^{\infty,2}}.$$

The first term on the right-hand side is now equivalent to the case  $l \leq k \leq m$ . The second term on the right-hand side is again of the same form as  $A$ .

The remaining case  $m \leq k$  and  $l \leq k$  is negligible, since due to Fourier supports, these vanish whenever both  $m$  and  $l$  are less than, say,  $k-5$ . Finally, the low-order versions of (3.18) and (3.19), i.e., those containing  $P_{\lesssim 0}$ , can be proved similarly, using the corresponding low-order versions of (3.4) and (3.9).  $\square$

**3.6. Improved Product Estimates.** We can now apply the envelope estimates (3.18) and (3.19) to obtain some “improved” bilinear product estimates. The improvement here refers to the fact that the right-hand sides of these inequalities contain only  $L_{t,x}^{2,2}$ -type norms, rather than the infinitesimally worse Besov norms found in the right-hand sides in Theorems 3.5 and 3.7. Again, some of these estimates were discussed in [13], so we keep the expositions brief here.

**Theorem 3.9.** *For any  $\phi, \psi \in C_t^\infty \mathcal{S}_x \mathcal{N}$ ,*

$$(3.20) \quad \delta \|\phi \cdot \psi\|_{B_{t,x}^{\infty,0}} \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1},$$

$$(3.21) \quad \delta \|\mathfrak{T}(\nabla_t \phi \cdot \psi)\|_{B_{t,x}^{\infty,0}} \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}.$$

*Proof.* For (3.20), we begin by decomposing

$$\begin{aligned} \delta \|\phi \psi\|_{B_{t,x}^0} &\lesssim \delta \sum_{k,l,m \geq 0} \|P_k(P_{\sim l} \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}} + \delta \sum_{k,l \geq 0} \|P_k(P_{\sim l} \phi \cdot P_{\leq 0} \psi)\|_{L_{t,x}^{\infty,2}} \\ &\quad + \delta \sum_{k,l \geq 0} \|P_k(P_{\leq 0} \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}} + \delta \|P_{\leq 0}(\phi \psi)\|_{L_{t,x}^{\infty,2}} \\ &= HH + HL + LH + L. \end{aligned}$$

The term  $L$  can be bounded trivially using (3.7):

$$L \lesssim \delta^{\frac{1}{2}} \|\phi\|_{L_{x,t}^{4,\infty}} \cdot \delta^{\frac{1}{2}} \|\psi\|_{L_{x,t}^{4,\infty}} \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}.$$

The remaining terms are controlled using (3.18). First,

$$HH \lesssim \sum_{k,l,m \geq 0} 2^{-\frac{1}{2}|k-l| - \frac{1}{2}|k-m|} N_l \phi N_m \psi \lesssim \sum_{k \geq 0} E_k \phi E_k \psi \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}.$$

In the last step, we applied Hölder’s inequality and (3.8). Similarly, we also have

$$\begin{aligned} HL &\lesssim \sum_{k,l,m \geq 0} 2^{-\frac{1}{2}|k-l| - \frac{1}{2}k} N_l \phi \|\psi\|_{H_{t,x}^1} \lesssim \sum_{k \geq 0} E_k \phi E_k \psi \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}, \\ LH &\lesssim \sum_{k,l,m \geq 0} 2^{-\frac{1}{2}k - \frac{1}{2}|k-m|} \|\phi\|_{H_{t,x}^1} N_m \psi \lesssim \sum_{k \geq 0} E_k \phi E_k \psi \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}. \end{aligned}$$

This completes the proof of (3.20).

The approach to proving (3.21) is analogous. We again decompose

$$\begin{aligned} \delta \|\mathfrak{T}(\nabla_t \phi \cdot \psi)\|_{B_{t,x}^0} &\lesssim HH + HL + LH + L, \\ HH &= \delta \sum_{k,l,m \geq 0} \|P_k \mathfrak{T}(P_{\sim l} \nabla_t \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}}, \\ HL &= \delta \sum_{k,l \geq 0} \|P_k \mathfrak{T}(P_{\sim l} \nabla_t \phi \cdot P_{\leq 0} \psi)\|_{L_{t,x}^{\infty,2}}, \\ LH &= \delta \sum_{k,l \geq 0} \|P_k \mathfrak{T}(P_{\leq 0} \nabla_t \phi \cdot P_{\sim m} \psi)\|_{L_{t,x}^{\infty,2}}, \\ L &= \delta \|P_{\leq 0} \mathfrak{T}(\nabla_t \phi \cdot \psi)\|_{L_{t,x}^{\infty,2}}. \end{aligned}$$

The low-order term  $L$  is controlled using (3.4):

$$L \lesssim \delta \|\nabla_t \phi \cdot \psi\|_{L_{x,t}^{1,1}} \lesssim \delta \|\nabla_t \phi\|_{L_{t,x}^{2,2}} \|\psi\|_{L_{t,x}^{2,2}} \leq \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}.$$

The nontrivial terms are now controlled using (3.8) and (3.19):

$$\begin{aligned}
HH &\lesssim \sum_{k,l,m \geq 0} 2^{-\frac{1}{2}|k-l|-\frac{1}{2}|k-m|} N_l \phi N_m \psi \lesssim \sum_{k \geq 0} E_k \phi E_k \psi \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}, \\
HL &\lesssim \sum_{k,l,m \geq 0} 2^{-\frac{1}{2}|k-l|-\frac{1}{2}k} N_l \phi \|\psi\|_{H_{t,x}^1} \lesssim \sum_{k \geq 0} E_k \phi E_k \psi \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}, \\
LH &\lesssim \sum_{k,l,m \geq 0} 2^{-\frac{1}{2}k-\frac{1}{2}|k-m|} \|\phi\|_{H_{t,x}^1} N_m \psi \lesssim \sum_{k \geq 0} E_k \phi E_k \psi \lesssim \|\phi\|_{H_{t,x}^1} \|\psi\|_{H_{t,x}^1}.
\end{aligned}$$

This process is completely analogous to that for proving (3.20).  $\square$

#### 4. GEOMETRIC ESTIMATES

We now return to the fully abstract setting of Section 2. Throughout the remainder of the paper, we will also impose the following additional conditions:

- *We will always assume that  $\mathcal{S}$  is compact.* In particular,  $\mathcal{S}$  has finite area with respect to any of the metrics  $\gamma[v]$ .
- For convenience, *we also normalize so that  $(\mathcal{S}, \gamma[0])$  has unit area.*<sup>29</sup>

Our next objective is to describe in detail the regularity assumptions we wish to impose on our system. After defining these assumptions, we establish some of their basic consequences, including frame and Sobolev estimates. At the end of this section, we will again relate the theory we have developed to our main setting of interest: regular null cones with finite curvature flux. In particular, we justify that the assumptions we made apply to this specific family of problems.

From now on, we let  $\mathbf{b}$ ,  $\mathbf{c}$ ,  $\mathbf{d}$ , and  $\mathbf{R}$  denote positive real constants, we let  $\mathbf{N}$  denote a positive integer constant, and we fix an exponent  $\mathbf{e} \in (2, \infty]$ .

**4.1. Evolutionary Bounds.** The first collection of assumptions we will need involves control on how the geometries of the  $\gamma[v]$ 's evolve. This is expressed using various weak integral bounds on the second fundamental form  $k = k^{(\gamma)}$ . The specific bounds we will repeatedly reference throughout the paper are listed below.

- We will almost always require the following bounds for  $k$ :<sup>30</sup>

$$(4.1) \quad \|\mathrm{tr} k\|_{L_{x,t}^{\infty,1}} \leq 2\mathbf{c},$$

$$(4.2) \quad \|k\|_{L_{x,t}^{\infty,1}} \leq \mathbf{c}.$$

- Later, we may also need control on one derivative of  $\mathrm{tr} k$ :

$$(4.3) \quad \|\nabla(\mathrm{tr} k)\|_{L_{x,t}^{2,1}} \leq \mathbf{b}.$$

- Another important condition is the following on  $\mathfrak{C}$ , where  $\mathbf{e} > 2$ :

$$(4.4) \quad \inf\{\|\Phi\|_{L_{x,t}^{\mathbf{e},\infty}} \mid \Phi \in \mathcal{C}^\infty \underline{\mathcal{T}}_3^0 \mathcal{N}, \quad \nabla_t \Phi = \mathfrak{C}\} \leq \mathbf{d}.$$

In other words, some covariant  $t$ -antiderivative of  $\mathfrak{C}$  has  $L_{x,t}^{\mathbf{e},\infty}$ -control.

Next, we discuss some basic consequences of some of the above conditions. First of all, the bound (4.2) implies uniform bounds for the Jacobian  $\mathcal{J} = \mathcal{J}^{(\mathbf{e})}$ .

**Proposition 4.1.** *If (4.1) holds, then we have the following uniform comparison:*

$$(4.5) \quad e^{-2\mathbf{c}} \leq \mathcal{J} \leq e^{2\mathbf{c}}.$$

<sup>29</sup>Of course, one can always reduce to this from general  $\gamma$  via a scaling argument.

<sup>30</sup>Note that (4.2) implies (4.1).

*Proof.* This is a trivial consequence of the definition of  $\mathcal{J}$ . □

**Corollary 4.2.** *If (4.1) holds, and if  $\mathcal{A}_v$  is the area of  $(\mathcal{S}, \gamma[v])$ , then*

$$(4.6) \quad e^{-2c} \leq \mathcal{A}_v \leq e^{2c}.$$

*Proof.* This follows immediately from (4.5). □

Next, we apply (4.5) to derive some basic integrated calculus estimates.

**Proposition 4.3.** *Assume (4.1), and fix  $q \in [1, \infty]$ . If  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , then*

$$(4.7) \quad \|\mathfrak{T}\Psi\|_{L_{x,t}^{q,\infty}} \lesssim^c \|\Psi\|_{L_{x,t}^{q,1}}.$$

*Proof.* First, assume  $q < \infty$ . Applying (2.4) along with (4.5), then

$$\begin{aligned} \|\mathfrak{T}\Psi\|_{L_{x,t}^{q,\infty}}^q &\leq e^{2c} \int_{\mathcal{S}} \left| \sup_{0 \leq v < \delta} \int_0^v |\Psi|_{(w,\omega)} dw \right|^q d\epsilon[0] \\ &\leq e^{2c} \int_{\mathcal{S}} \left( \int_0^\delta |\Psi|_{(v,\omega)} dv \right)^q d\epsilon[0] \\ &\leq e^{4c} \|\Psi\|_{L_{x,t}^{q,1}}^q. \end{aligned}$$

The remaining case  $q = \infty$  is proved similarly:

$$\|\mathfrak{T}\Psi\|_{L_{x,t}^{\infty,\infty}} \leq \sup_{\omega \in \mathcal{S}} \int_0^\delta |\Psi|_{(w,\omega)} dw = \|\Psi\|_{L_{x,t}^{\infty,1}}. \quad \square$$

**Corollary 4.4.** *Fix  $p_1, p_2, q \in [1, \infty]$  and  $q_1, q_2 \in [q, \infty]$ , with*

$$q_1^{-1} + q_2^{-1} = q^{-1}, \quad p_1^{-1} + p_2^{-1} = 1.$$

*Assume that (4.1) holds. If  $\Psi_i \in \mathcal{C}^\infty \underline{T}_{l_i}^{r_i} \mathcal{N}$ , where  $i \in \{1, 2\}$ , then*

$$(4.8) \quad \|\mathfrak{T}(\Psi_1 \otimes \Psi_2)\|_{L_{x,t}^{q,\infty}} \lesssim^c \|\Psi_1\|_{L_{x,t}^{q_1,p_1}} \|\Psi_2\|_{L_{x,t}^{q_2,p_2}}.$$

*Proof.* This follows trivially from (4.7) and Hölder's inequality. □

Finally, we prove a basic estimate for the derivative of the Jacobian.

**Proposition 4.5.** *If (4.2) holds, and if  $q \in [1, \infty]$ , then*

$$(4.9) \quad \|\nabla \mathcal{J}\|_{L_{x,t}^{q,\infty}} \lesssim^c \|\nabla(\text{tr } k)\|_{L_{x,t}^{q,1}}.$$

*Proof.* We begin by applying (4.5) to obtain

$$\|\nabla \mathcal{J}\|_{L_{x,t}^{q,\infty}} = \|\mathcal{J} \cdot \nabla \mathfrak{T}(\text{tr } k)\|_{L_{x,t}^{q,\infty}} \lesssim^c \|\nabla \mathfrak{T}(\text{tr } k)\|_{L_{x,t}^{q,\infty}}.$$

For any  $(v, \omega) \in \mathcal{N}$ , we have from (2.4) and (2.5) that

$$|\nabla \mathfrak{T}(\text{tr } k)|_{(v,\omega)} \leq \int_0^\delta |\nabla(\text{tr } k)|_{(w,\omega)} dw + \int_0^v |k| |\nabla \mathfrak{T}(\text{tr } k)|_{(w,\omega)} dw.$$

By the Grönwall inequality, then

$$|\nabla \mathfrak{T}(\text{tr } k)|_{(v,\omega)} \leq (\exp \|k\|_{L_{x,t}^{\infty,1}}) \cdot \int_0^\delta |\nabla(\text{tr } k)|_{(w,\omega)} dw.$$

Taking an  $L_{x,t}^{q,\infty}$ -norm of the above, then we obtain

$$\|\nabla \mathfrak{T}(\text{tr } k)\|_{L_{x,t}^{q,\infty}} \leq e^c \|\nabla(\text{tr } k)\|_{L_{x,t}^{q,1}}.$$

Combining the above completes the proof of (4.9). □

**4.2. Frame Estimates.** The next task is to apply the assumptions (4.1)-(4.4) to derive estimates for equivariant and  $t$ -parallel fields. Later, we will apply these estimates in order to control families of equivariant and  $t$ -parallel horizontal frames. Throughout, we will let  $U$  denote an open subset of  $\mathcal{S}$ .

We begin first with equivariant fields. These are essentially the estimates resulting from the “weak regularity” condition used in [11, 13, 17, 18, 22, 23].

**Proposition 4.6.** *If  $Z \in \mathcal{C}^\infty \underline{T}_t^r \mathcal{N}_U$  is equivariant, and if (4.2) holds, then*

$$(4.10) \quad \|Z\|_{L_{t,x}^{\infty,\infty}} \leq e^{(r+l)c} \|Z[0]\|_{L_x^\infty}.$$

*Proof.* Since  $\mathfrak{L}_t Z \equiv 0$ , then  $\nabla_t Z$  is the sum of  $r+l$  terms, each of the form  $k \otimes Z$ , with one  $\gamma$ -contraction. Thus, by (2.4), if  $(v, \omega) \in \mathcal{N}$ , then

$$\begin{aligned} |Z|_{(v,\omega)} &\leq |Z|_{(0,\omega)} + \mathfrak{T}(|\nabla_t Z|)_{(v,\omega)} \\ &\leq |Z|_{(0,\omega)} + (r+l) \int_0^v |k| |Z|_{(w,\omega)} dw. \end{aligned}$$

Applying Grönwall’s inequality to the above yields

$$|Z|_{(v,\omega)} \leq |Z|_{(0,\omega)} \exp[(r+l)\|k\|_{L_{\omega,t}^{\infty,1}}] \leq e^{(r+l)c} |Z|_{(0,\omega)}.$$

Our desired estimate (4.10) now follows immediately from the above.  $\square$

We will also need an estimate for the covariant derivative of a equivariant field. Just as the zero-order estimate (4.10) required some control for  $k$ , a corresponding first-order estimate will require analogous control for the derivative of  $k$ .

**Proposition 4.7.** *If  $Z \in \mathcal{C}^\infty \underline{T}_t^r \mathcal{N}_U$  is equivariant, and if (4.2) holds, then for any  $q \in [2, \infty]$ , we have the following estimate:*

$$(4.11) \quad \|\nabla Z\|_{L_{x,t}^{q,\infty}} \lesssim^{c,(r+l)c} \|\nabla Z[0]\|_{L_x^q} + (r+l)\|Z[0]\|_{L_x^\infty} \|\nabla k\|_{L_{x,t}^{q,1}}.$$

*Proof.* By the commutation formula (2.3), we have the estimate

$$|\nabla_t \nabla Z| \leq |\nabla \nabla_t Z| + |k| |\nabla Z| + (r+l) |\mathfrak{C}| |Z|.$$

Recalling the form of  $\nabla_t Z$  in the proof of Proposition 4.6, and noting the crude estimate  $|\mathfrak{C}| \leq 2|\nabla k|$ , then we have derived the bound

$$|\nabla_t \nabla Z| \leq (r+l+1)|k| |\nabla Z| + 3(r+l) |Z| |\nabla k|.$$

As a result of the above and of (2.4), if  $(v, \omega) \in \mathcal{N}$ , then

$$\begin{aligned} |\nabla Z|_{(v,\omega)} &\leq |\nabla Z|_{(0,\omega)} + 3(r+l) \|Z\|_{L_{t,x}^{\infty,\infty}} \int_0^v |\nabla k|_{(w,\omega)} dw \\ &\quad + (r+l+1) \int_0^v |k| |\nabla Z|_{(w,\omega)} dw. \end{aligned}$$

Applying (4.10) and Gronwall’s inequality yields

$$\sup_{0 \leq v < \delta} |\nabla Z|_{(v,\omega)} \lesssim^{c,(r+l)c} |\nabla Z|_{(0,\omega)} + 3(r+l) \|Z[0]\|_{L_x^\infty} \int_0^\delta |\nabla k|_{(w,\omega)} dw.$$

Taking an  $L^q$ -norm of the above over  $\mathcal{S}$  and recalling (4.5) completes the proof.  $\square$

**Remark.** *We can only expect  $L_{t,x}^{2,2}$ -type bounds for  $\nabla k$  in our motivational problem of regular null cones with bounded curvature flux. Thus, by applying Proposition 4.7, we can only obtain  $L_{x,t}^{2,\infty}$ -type estimates for  $\nabla Z$ .*

We now present estimates analogous to Propositions 4.6 and 4.7, but for  $t$ -parallel fields. In particular, we will examine how one can potentially achieve strictly better estimates in this case. First of all, if  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}_U$  is  $t$ -parallel, then  $\nabla_t |\Psi|^2 \equiv 0$ , and hence  $|\Psi|$  is constant with respect to  $t$ . This implies the following identity.

**Proposition 4.8.** *If  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}_U$  is  $t$ -parallel, then*

$$(4.12) \quad \|\Psi\|_{L_{t,x}^{\infty,\infty}} = \|\Psi[0]\|_{L_x^\infty}.$$

On the other hand, since  $\nabla \Psi$  is no longer  $t$ -parallel, then the above is no longer true for covariant derivatives of  $\Psi$ . However, given some control for  $k$ , we can still provide some control for  $\nabla \Psi$ . One example is the following proposition.

**Proposition 4.9.** *Suppose  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}_U$  is  $t$ -parallel, and assume (4.2) holds. If  $q \in [1, \infty]$ , and if  $\Phi \in \mathcal{C}^\infty \underline{T}_3^0 \mathcal{N}$  is such that  $\nabla_t \Phi = \mathfrak{C}$ , then*

$$(4.13) \quad \|\nabla \Psi\|_{L_{x,t}^{q,\infty}} \lesssim^c \|\nabla \Psi[0]\|_{L_x^q} + (r+l) \|\Psi[0]\|_{L_x^\infty} \|\Phi\|_{L_{x,t}^{q,\infty}}.$$

*Proof.* Like in the proof of Proposition 4.7, we apply (2.3) and (2.4) to obtain

$$|\nabla \Psi|_{(v,\omega)} \lesssim |\nabla \Psi|_{(0,\omega)} + |\mathfrak{T}(k \otimes \nabla \Psi)|_{(v,\omega)} + (r+l) |\mathfrak{T}(\mathfrak{C} \otimes \Psi)|_{(v,\omega)}$$

for any  $(v, \omega) \in \mathcal{N}$ . Since

$$\mathfrak{T}(\mathfrak{C} \otimes \Psi) = \mathfrak{T}(\nabla_t \Phi \otimes \Psi) = \Phi \otimes \Psi - \mathfrak{p}\{(\Phi \otimes \Psi)[0]\},$$

where we integrated by parts and recalled that  $\Psi$  is  $t$ -parallel, then

$$\begin{aligned} |\nabla \Psi|_{(v,\omega)} &\lesssim |\nabla \Psi|_{(0,\omega)} + \int_0^v |k| |\nabla \Psi|_{(w,\omega)} dw + (r+l) \|\Psi\|_{L_{t,x}^{\infty,\infty}} \sup_{0 \leq w < \delta} |\Phi|_{(w,\omega)} \\ &\lesssim |\nabla \Psi|_{(0,\omega)} + (r+l) \|\Psi[0]\|_{L_x^\infty} \sup_{0 \leq w < \delta} |\Phi|_{(w,\omega)} + \int_0^v |k| |\nabla \Psi|_{(w,\omega)} dw. \end{aligned}$$

Applying Grönwall's inequality to the above yields

$$\sup_{0 \leq v < \delta} |\nabla \Psi|_{(v,\omega)} \lesssim e^c [|\nabla \Psi|_{(0,\omega)} + (r+l) \|\Psi[0]\|_{L_x^\infty} \sup_{0 \leq v < \delta} |\Phi|_{(v,\omega)}].$$

Taking an  $L^q$ -norm of this over  $\mathcal{S}$  and recalling (4.5) proves (4.13).  $\square$

In summary, the estimate (4.11) depends on the regularity of  $\nabla k$ , while (4.13) depends only on  $\mathfrak{C}$ . Thus, if  $\mathfrak{C}$  has better bounds than  $\nabla k$ , then Proposition 4.9 may yield strictly better control than Proposition 4.7.

**Remark.** *In our motivational problem of regular null cones with bounded curvature flux, we can find  $\Phi$  satisfying the hypotheses of Proposition 4.9 which has good  $L_{x,t}^{4,\infty}$ -control. Thus, Proposition 4.9 yields strictly better estimates for  $t$ -parallel frames than Proposition 4.7 does for equivariant frames.*

**4.3. Initial Regularity.** Next, we quantify how several of the estimates we will derive in our analyses will depend on the “initial” surface  $(\mathcal{S}, \gamma[0])$ . For this purpose, we compact all the quantitative descriptions into a single condition.

We say that the condition **(I)** (or **(I)**<sub>e,N,R</sub>, to be more specific), holds iff:

- $\mathcal{S}$  can be covered by  $N$  local coordinate systems  $(U_i, \varphi_i)$ ,  $1 \leq i \leq N$ , with each  $\varphi_i(U_i)$  being a bounded neighborhood in  $\mathbb{R}^2$ .

- For any  $1 \leq i \leq \mathbf{N}$ , indexing with respect to the  $\varphi_i$ -coordinate system on  $U_i$ , then we have the uniform positivity property

$$(4.14) \quad \frac{1}{2}|\xi|^2 \leq \sum_{a,b=1}^2 (\gamma[0])_{ab} \xi^a \xi^b \leq 2|\xi|^2, \quad \xi \in \mathbb{R}^2.$$

Furthermore, if  $\partial_1^{(i)}, \partial_2^{(i)}$  are the  $\varphi_i$ -coordinate vector fields, then

$$(4.15) \quad \|\nabla \partial_a^{(i)}\|_{L_x^2(\gamma[0])} \leq R, \quad a \in \{1, 2\}.$$

- On each  $1 \leq i \leq \mathbf{N}$ , there exists an orthonormal frame  $e_1^{(i)}, e_2^{(i)}$  on  $U_i$  which satisfies in addition (recall that  $e > 2$ ) the connection estimate

$$(4.16) \quad \|\nabla e_a^{(i)}\|_{L_x^e(\gamma[0])} \leq R, \quad a \in \{1, 2\}.$$

- There is a partition of unity  $\{\eta_i \mid 1 \leq i \leq \mathbf{N}\}$  of  $\mathcal{S}$ , subordinate to the  $U_i$ 's, such that for each  $1 \leq i \leq \mathbf{N}$ , we have

$$(4.17) \quad 0 \leq \eta_i \leq 1, \quad |\partial_a^{(i)} \eta_i| \leq R, \quad |\partial_a^{(i)} \partial_b^{(i)} \eta_i| \leq R, \quad a, b \in \{1, 2\}.$$

- Furthermore, for each  $1 \leq i \leq \mathbf{N}$ , there is map  $\tilde{\eta}_i \in C^\infty \mathcal{S}$ , supported within  $U_i$  and identically 1 on the support of  $\eta_i$ , such that

$$(4.18) \quad 0 \leq \tilde{\eta}_i \leq 1, \quad |\partial_a^{(i)} \tilde{\eta}_i| \leq R, \quad a \in \{1, 2\}.$$

Note that since  $\mathcal{S}$  is compact, then **(I)** always holds for some  $\mathbf{N}$  and  $R$ , given any  $e \in (2, \infty]$ . For example, one can generate the coordinate systems  $(U_i, \varphi_i)$  by taking sufficiently small normal coordinate systems about a point. The orthonormal frame can then be constructed by parallel transporting an orthonormal basis of vectors at the same point along all radial geodesics. In other words, in our setting, the **(I)** condition is in a sense trivial, as it only serves to better describe how the main estimates of this paper depend on the initial surface.

In addition to the above, we also define for each  $1 \leq i \leq \mathbf{N}$  the function

$$(4.19) \quad \vartheta_i \in C^\infty U_i, \quad \vartheta_i = \sqrt{|(\gamma[0])_{11}(\gamma[0])_{22} - (\gamma[0])_{12}^2|},$$

where we have indexed  $\gamma[0]$  with respect to the  $\varphi_i$ -coordinates. These quantities  $\vartheta_i$  are present within change-of-variables formulas involving  $(\mathcal{S}, \gamma[0])$  and  $\mathbb{R}^2$ :

$$\int_{U_i} f \cdot d\epsilon[0] = \int_{\varphi_i(U_i)} [(\vartheta_i f) \circ \varphi_i^{-1}], \quad f \in C^\infty U_i.$$

We will need to refer to these  $\vartheta_i$ 's in Section 6.

We finish here with some basic propagation of regularity properties for the  $\eta_i$ 's,  $\tilde{\eta}_i$ 's, and  $\vartheta_i$ 's associated with our **(I)** assumption.

**Proposition 4.10.** *Assume (4.2) and **(I)**. Moreover, let  $1 \leq i \leq \mathbf{N}$ , and suppose  $U_i, \varphi_i, \eta_i, \tilde{\eta}_i$  are as in the above definition of the **(I)** condition. Then,*<sup>31</sup>

$$(4.20) \quad \|\nabla(\epsilon \eta_i)\|_{L_{t,x}^{\infty,\infty}} \lesssim^c R, \quad \|\nabla(\epsilon \tilde{\eta}_i)\|_{L_{t,x}^{\infty,\infty}} \lesssim^c R.$$

Moreover, with  $\vartheta_i$  as defined in (4.19), we have

$$(4.21) \quad \|\epsilon \vartheta_i\|_{L_{t,x}^{\infty,\infty}} \lesssim 1, \quad \|\nabla(\epsilon \vartheta_i)\|_{L_{t,x}^{\infty,\infty}} \lesssim^c R.$$

<sup>31</sup>Recall that  $\epsilon \eta_i$  and  $\epsilon \tilde{\eta}_i$  denote the equivariant transports of  $\eta_i$  and  $\tilde{\eta}_i$ , respectively.



*Proof.* First of all, by (4.14) and (4.17), we have the bounds

$$\|\nabla(\mathfrak{e}\eta_i)[0]\|_{L_x^\infty} \lesssim R, \quad \|\nabla(\mathfrak{e}\tilde{\eta}_i)[0]\|_{L_x^\infty} \lesssim R.$$

Since  $\mathfrak{e}\eta_i$  and  $\mathfrak{e}\tilde{\eta}_i$  are both equivariant and  $t$ -parallel, then we can apply either of the estimates (4.11) or (4.13) in order to obtain (4.20). The bounds in (4.21) follow immediately from (4.14) and a similar argument as above.  $\square$

Similar propagation estimates can also be made for the frame elements  $\partial_a^{(i)}$  and  $e_a^{(i)}$  in the definition of the **(I)** condition; see Section 5.1.

**4.4. Sobolev Estimates.** A well-known application of the **(I)** condition is deriving first-order scalar Sobolev estimates on  $(\mathcal{S}, \gamma[0])$ .<sup>32</sup> Indeed, this can be done by applying the Euclidean Sobolev bounds on each coordinate system  $(U_i, \varphi_i)$ .

**Proposition 4.11.** *Let  $\phi \in C^\infty \mathcal{N}$ , and suppose **(I)** holds.*

- *The following Gagliardo-Nirenberg-Sobolev inequality holds:*

$$(4.22) \quad \|\phi[0]\|_{L_x^2} \lesssim_{\mathbf{N}, \mathbf{R}} \|\nabla \phi[0]\|_{L_x^1} + \|\phi[0]\|_{L_x^1}.$$

- *The following Gagliardo-Nirenberg inequality holds for any  $q \in (2, \infty)$ :*

$$(4.23) \quad \|\phi[0]\|_{L_x^\infty} \lesssim_{\mathbf{N}, \mathbf{R}, q} \|\nabla \phi[0]\|_{L_x^q}^{\frac{2}{q}} \|\phi[0]\|_{L_x^q}^{1-\frac{2}{q}} + \|\phi[0]\|_{L_x^q}.$$

If we combine the **(I)** condition with the assumption (4.2), then we can prove corresponding *uniform* Sobolev inequalities on all the  $\mathcal{S}_v$ 's, *independently of  $v$* . Inequalities of this type were proved in [12, 16], for example, directly from coordinate decompositions and (4.10). Here, we present a similar proof using (4.13).

**Proposition 4.12.** *Assume both (4.2) and **(I)**, and fix  $v \in [0, \delta)$ .*

- *If  $\phi \in C^\infty \mathcal{N}$ , then*

$$(4.24) \quad \|\phi[v]\|_{L_x^2} \lesssim_{\mathbf{N}, \mathbf{R}} \|\nabla \phi[v]\|_{L_x^1} + \|\phi[v]\|_{L_x^1},$$

- *If  $q \in (2, \infty)$  and  $\phi \in C^\infty \mathcal{N}$ , then*

$$(4.25) \quad \|\phi[v]\|_{L_x^\infty} \lesssim_{\mathbf{N}, \mathbf{R}, q} \|\nabla \phi[v]\|_{L_x^q}^{\frac{2}{q}} \|\phi[v]\|_{L_x^q}^{1-\frac{2}{q}} + \|\phi[v]\|_{L_x^q}.$$

*Proof.* Let  $\tilde{\phi} = \mathfrak{e}(\phi[v]) = \mathfrak{p}(\phi[v])$ , that is,  $\tilde{\phi} \in C^\infty \mathcal{N}$  is defined to be the  $t$ -parallel transport of  $\phi[v]$ . First, by (4.5) and (4.22), we have

$$\begin{aligned} \|\phi[v]\|_{L_x^2} &\lesssim^c \left( \int_{\mathcal{S}} |\tilde{\phi}|^2|_{(0, \omega)} d\epsilon[0]_\omega \right)^{\frac{1}{2}} \\ &\lesssim_{\mathbf{N}, \mathbf{R}} \int_{\mathcal{S}} |\nabla \tilde{\phi}|_{(0, \omega)} d\epsilon[0]_\omega + \int_{\mathcal{S}} |\tilde{\phi}|_{(0, \omega)} d\epsilon[0]_\omega. \end{aligned}$$

By (4.5) and the definition of  $\tilde{\phi}$ , then

$$\int_{\mathcal{S}} |\tilde{\phi}|_{(0, \omega)} d\epsilon[0]_\omega \lesssim^c \int_{\mathcal{S}} |\phi|_{(v, \omega)} d\epsilon[v]_\omega = \|\phi[v]\|_{L_x^1}.$$

Moreover, applying (4.13) to  $\tilde{\phi}$  yields<sup>33</sup>

$$\int_{\mathcal{S}} |\nabla \tilde{\phi}|_{(0, \omega)} d\epsilon[0]_\omega \lesssim^c \int_{\mathcal{S}} |\nabla \phi|_{(v, \omega)} d\epsilon[v]_\omega \lesssim^c \|\nabla \phi[v]\|_{L_x^1}.$$

This completes the proof of (4.24); the proof of (4.25) is analogous.  $\square$

<sup>32</sup>In fact, we need only a small portion of the **(I)** condition for this.

<sup>33</sup>Here, we apply (4.13) to the adjusted foliation  $\mathcal{S}'_{v'} = \mathcal{S}_{v-v'}$ .

Next, we will apply Proposition 4.12 in order to prove similar Sobolev estimates for *tensorial* quantities. This is trickier, since the connection  $\nabla$  now depends on the metric  $\gamma$ . However, we can sidestep this issue by dealing with the square norm of the tensor field, which is scalar, and recalling that  $\gamma$  and  $\nabla$  are compatible.

**Proposition 4.13.** *Assume both (4.2) and (I), and fix  $v \in [0, \delta)$ .*

- *If  $q \in (2, \infty)$  and  $\Psi \in \mathcal{C}^\infty \underline{T}_I^r \mathcal{N}$ , then*

$$(4.26) \quad \|\Psi[v]\|_{L_x^q} \lesssim_{\mathbf{N}, \mathbf{R}, q}^c \|\nabla \Psi[v]\|_{L_x^2}^{1-\frac{2}{q}} \|\Psi[v]\|_{L_x^2}^{\frac{2}{q}} + \|\Psi[v]\|_{L_x^2},$$

$$(4.27) \quad \|\Psi[v]\|_{L_x^\infty} \lesssim_{\mathbf{N}, \mathbf{R}, q}^c \|\nabla \Psi[v]\|_{L_x^q}^{\frac{2}{q}} \|\Psi[v]\|_{L_x^q}^{1-\frac{2}{q}} + \|\Psi[v]\|_{L_x^q}.$$

- *If  $\Psi \in \mathcal{C}^\infty \underline{T}_I^r \mathcal{N}$ , then*

$$(4.28) \quad \|\Psi[v]\|_{L_x^\infty} \lesssim_{\mathbf{N}, \mathbf{R}}^c \|\nabla^2 \Psi[v]\|_{L_x^2}^{\frac{1}{2}} \|\Psi[v]\|_{L_x^2}^{\frac{1}{2}} + \|\Psi[v]\|_{L_x^2}.$$

*Proof.* <sup>34</sup> First, we make use of (4.24) as follows:

$$\begin{aligned} \|\Psi[v]\|_{L_x^q}^{\frac{q}{2}} &\lesssim_{\mathbf{N}, \mathbf{R}}^c \|\nabla |\Psi|^{\frac{q}{2}}[v]\|_{L_x^1} + \| |\Psi|^{\frac{q}{2}}[v] \|_{L_x^1} \\ &\lesssim_q (\|\nabla \Psi[v]\|_{L_x^2} + \|\Psi[v]\|_{L_x^2}) \|\Psi[v]\|_{L_x^{\frac{q-2}{2}}}^{\frac{q-2}{2}}. \end{aligned}$$

Applying the above with  $q = 2k$ , with  $k > 1$  an integer, yields

$$\|\Psi[v]\|_{L_x^{2k}}^k \lesssim_{\mathbf{N}, \mathbf{R}, k}^c (\|\nabla \Psi[v]\|_{L_x^2} + \|\Psi[v]\|_{L_x^2}) \|\Psi[v]\|_{L_x^{2k-2}}^{k-1}.$$

Taking  $k = 2, 3, 4, \dots$  and applying an induction argument, we obtain

$$\|\Psi[v]\|_{L_x^{2k}} \lesssim_{\mathbf{N}, \mathbf{R}, k}^c (\|\nabla \Psi[v]\|_{L_x^2} + \|\Psi[v]\|_{L_x^2})^{\frac{k-1}{k}} \|\Psi[v]\|_{L_x^2}^{\frac{1}{k}}.$$

This is (4.26) when  $q$  is an even integer; the general case follows via interpolation.

The idea for proving (4.27) is similar. We apply (4.25) as follows:

$$\begin{aligned} \|\Psi[v]\|_{L_x^\infty}^2 &\lesssim_{\mathbf{N}, \mathbf{R}, q}^c \|\nabla |\Psi|^2[v]\|_{L_x^q}^{\frac{2}{q}} \| |\Psi|^2[v] \|_{L_x^q}^{1-\frac{2}{q}} + \| |\Psi|^2[v] \|_{L_x^q} \\ &\lesssim \|\Psi[v]\|_{L_x^\infty} (\|\nabla \Psi[v]\|_{L_x^q}^{\frac{2}{q}} \|\Psi[v]\|_{L_x^q}^{1-\frac{2}{q}} + \|\Psi[v]\|_{L_x^q}). \end{aligned}$$

The inequality (4.27) follows immediately. Finally, for (4.28), we apply (4.27) and (4.26) in succession and integrate by parts to eliminate the  $L^2$ -norms of  $\nabla \Psi$ .  $\square$

We also take a brief look at tensorial Sobolev inequalities involving all of  $\mathcal{N}$ . These are the analogues of Proposition 3.1 in our geometric setting.

**Proposition 4.14.** *Let  $\Psi \in \mathcal{C}^\infty \underline{T}_I^r \mathcal{N}$ .*

- *If (4.1) holds, then*

$$(4.29) \quad \delta^{\frac{1}{2}} \|\Psi\|_{L_{x,t}^{2,\infty}} \lesssim^c (\delta \|\nabla_t \Psi\|_{L_{t,x}^{2,2}} + \|\Psi\|_{L_{t,x}^{2,2}})^{\frac{1}{2}} \|\Psi\|_{L_{t,x}^{2,2}}^{\frac{1}{2}}.$$

- *If both (4.2) and (I) hold, then*

$$(4.30) \quad \delta^{\frac{1}{2}} \|\Psi\|_{L_{x,t}^{4,\infty}} \lesssim_{\mathbf{N}, \mathbf{R}}^c (\delta \|\nabla_t \Psi\|_{L_{t,x}^{2,2}} + \|\Psi\|_{L_{t,x}^{2,2}})^{\frac{1}{2}} (\|\nabla \Psi\|_{L_{t,x}^{2,2}} + \|\Psi\|_{L_{t,x}^{2,2}})^{\frac{1}{2}}.$$

---

<sup>34</sup>The proof of (4.26) is identical to that of [12, Cor. 2.4]; we reproduce it here for convenience.

The proofs of (4.29) and (4.30), which we omit here, are nearly identical to their Euclidean analogues (3.6) and (3.7). The main difference is that rather than applying the standard Sobolev estimates for  $\mathbb{R}^2$ , here we use the preceding geometric analogues.<sup>35</sup> Moreover, since  $\nabla$  and  $\nabla_t$  annihilate  $\gamma$ , then the partial derivatives in the Euclidean versions naturally turn into covariant derivatives here. For some further details on this process, see, e.g., [22, Lemma 3.2].

**4.5. Null Cones with Finite Curvature Flux.** We now return to the specific setting of a geodesically foliated spherical null cone within a vacuum spacetime, as described at the end of the Section 2. In particular, we let  $\mathcal{N}$ ,  $t$ , and  $\gamma$  be as before. Furthermore, we impose similar assumptions as in [11]: that  $\delta = 1$ , and the curvature flux is small. More specifically, the second assumption is

$$(4.31) \quad \|\alpha\|_{L_{t,x}^{2,2}} + \|\beta\|_{L_{t,x}^{2,2}} + \|\rho\|_{L_{t,x}^{2,2}} + \|\sigma\|_{L_{t,x}^{2,2}} + \|\rho\|_{L_{t,x}^{2,2}} \leq C,$$

where  $C$  is some sufficiently small constant.<sup>36</sup> We must also assume that certain quantities depending on the Ricci coefficients have similarly small values on  $\mathcal{S}_0$ . We will not expand this point here; for details, see [11].

Under the above assumptions, then one has some associated bounds for the Ricci coefficients  $\chi$ ,  $\zeta$ , and  $\underline{\chi}$ . Some examples of these bounds are as follows:

$$(4.32) \quad \begin{aligned} \|\chi - r^{-1}\gamma\|_{H_{t,x}^1} + \|\chi - r^{-1}\gamma\|_{L_{x,t}^{\infty,2}} &\leq \Delta, \\ \|\zeta\|_{H_{t,x}^1} + \|\zeta\|_{L_{x,t}^{\infty,2}} &\leq \Delta, \\ \|\underline{\chi} + r^{-1}\gamma\|_{L_{x,t}^{2,\infty}} + \|\nabla_t(\underline{\chi} + r^{-1}\gamma)\|_{L_{t,x}^{2,2}} &\leq \Delta. \end{aligned}$$

Here,  $r \in C^\infty \mathcal{N}$  maps each  $(v, \omega) \in \mathcal{N}$  to the “radius” of  $\mathcal{S}_v$ , i.e.,

$$4\pi \cdot r^2|_{(v,\omega)} = \int_{\mathcal{S}} d\epsilon[v].$$

The constant  $\Delta$  in (4.32) is some positive number related to  $C$ .<sup>37</sup>

One generally reaches the estimates (4.32) in two ways:

- Under appropriate conditions, these bounds are consequences of (4.31). For example, these are some of the main estimates proved in [11].
- These bounds can also be used as “bootstrap assumptions”. This was the overall strategy adopted in [11] in order to prove the main results.<sup>38</sup>

The regularity assumptions (4.1)-(4.4) and their consequences reveal why the bootstrap assumptions mentioned above are necessary in the analysis in [11]. Indeed, in order to make use of many of the calculus estimates required in this analysis, one had to first show that such regularity conditions hold on  $\mathcal{N}$ . That these conditions hold follows only from appropriate bootstrap assumptions, including (4.32).

In the remainder of this section, we briefly sketch how the abstract assumptions used throughout this section follow from (4.32). In other words, we will justify that the results proved in this paper apply to our intended setting.

<sup>35</sup>We must also make use of the comparison (4.5).

<sup>36</sup>One can also consider the equivalent case of bounded but possibly large curvature flux, as long as the size  $\delta$  of the null cone segment is sufficiently small with respect to the flux; see [18].

<sup>37</sup>In the setting of [11], this  $\Delta$  must be sufficiently small, but large with respect to  $C$ .

<sup>38</sup>This is an application of the standard continuity argument, in which one assumes the desired estimates and proceeds to prove a strictly improved version of the same estimate.

First of all, the assumptions on  $\chi$  in the first line of (4.32) along with the observation (2.7) immediately imply the estimates (4.1)-(4.3).<sup>39</sup> Moreover, since  $\mathcal{S}$  is compact, as it is diffeomorphic to  $\mathbb{S}^2$ , then the condition **(I)** holds, with  $\mathbf{e} = 4$ , for some constants  $\mathbf{N}$  and  $\mathbf{R}$  depending on the geometry of  $\mathcal{S}_0$ .

Finally, we consider the condition (4.4), which is the new observation not applied in previous works [11, 17, 18, 22, 23] on various null cone settings. Recall that since  $k = \chi$ , then  $\mathfrak{C}$  is given precisely by the second identity in (2.10). Combining this with the second evolution equation in (2.8), we see that

$$\mathfrak{T}\mathfrak{C} \sim \zeta + \xi,$$

where  $\xi$  represents “lower-order” terms obtained by integrating the remaining terms in the above structure equations. From the bootstrap conditions and (4.30),

$$\|\zeta\|_{L_{x,t}^{4,\infty}} \lesssim \Delta.$$

On top of this, with a bit of extra work, then the “lower-order” terms  $\xi$  can also be controlled in the  $H_{t,x}^1$ -norm, so again by (4.30),

$$\|\xi\|_{L_{x,t}^{4,\infty}} \lesssim \Delta.$$

As a consequence of the above sketch of the argument, we have

$$\|\mathfrak{T}\mathfrak{C}\|_{L_{x,t}^{4,\infty}} \lesssim \Delta,$$

proving (4.4). This validates the use of Propositions 4.8 and 4.9 in the analysis, which forms the foundations of the scalar reduction presented in the next section.

## 5. SCALAR REDUCTIONS

In the Sobolev-type inequalities of Proposition 4.13, we reduced estimates for a horizontal tensorial quantity  $\Psi$  to one for a scalar quantity by dealing with  $|\Psi|^2$  instead. This allowed us to take advantage of the compatibility of  $\nabla$  with  $\gamma$  and hence avoid dealing with connection quantities resulting from a choice of frames. However, this method will not suffice for some situations, including some comparisons of derivative norms and certain Besov-type estimates for tensor fields.

In this section, we discuss the reduction of tensorial quantities to localized scalar analogues via  $t$ -parallel frames. Using this method in conjunction with the assumptions in Section 4, we can prove our main bilinear product estimates directly from their Euclidean analogues. In particular, we can bypass a vast majority of the technical machinery required in [13] associated with the heat flow, various commutator relations and estimates, and Gauss curvature estimates.

**5.1. Parallel Bases.** Throughout this section, we assume that the **(I)** condition holds on the initial manifold  $(\mathcal{S}, \gamma[0])$ . Furthermore, we will assume all the objects associated with this condition, as defined in Section 4.3. For instance, given any  $1 \leq i \leq \mathbf{N}$ , then  $(U_i, \varphi_i)$  denotes one of the coordinate systems associated with the **(I)** condition, and the symbols  $e_1^{(i)}, e_2^{(i)} \in \mathcal{C}^\infty T_0^1 \mathcal{S}$  refer to the corresponding orthonormal frame on  $U_i$ . For each such  $i$ , we also let  $e_{(i)}^1, e_{(i)}^2 \in \mathcal{C}^\infty T_1^0 \mathcal{S}$  denote the orthonormal coframe that is dual to the  $e_a^{(i)}$ 's.

Define  $\mathcal{X}_l^T(i)$  to be the collection of all horizontal fields of the form

$$\mathfrak{p}[e_{a_1}^{(i)} \otimes \cdots \otimes e_{a_l}^{(i)} \otimes e_{*(i)}^{b_1} \otimes \cdots \otimes e_{*(i)}^{b_r}] \in \mathcal{C}^\infty \underline{T}_r^l \mathcal{N}_{U_i},$$

---

<sup>39</sup>In particular, we have  $\mathbf{b} + \mathbf{c} \lesssim \Delta$ .

where  $1 \leq i \leq \mathbf{N}$  and  $a_1, \dots, a_l, b_1, \dots, b_r \in \{1, 2\}$ . This family  $\mathcal{X}_l^r(i)$  consists of exactly  $2^{r+l}$   $t$ -parallel fields of rank  $(l, r)$ , i.e., of rank dual to  $(r, l)$ . Moreover, by the **(I)** assumption, then each  $X \in \mathcal{X}_l^r(i)$  satisfies

$$(5.1) \quad \|X\|_{L_{t,x}^{\infty,\infty}} \leq 1, \quad \|\nabla X[0]\|_{L_x^e} \leq (r+l)\mathbf{R}.$$

**Remark.** Here, the definitions of the  $\mathcal{X}_l^r(i)$ 's depend entirely on assuming the **(I)** condition. The point, though, is that the relevant estimates involving these  $\mathcal{X}_l^r(i)$ 's will depend only on  $\mathbf{N}$  and  $\mathbf{R}$ , in addition to other constants in our assumptions.

**Remark.** As a special case, we can define  $\mathcal{X}_0^0(i)$  to be the set containing only the constant function with value 1 on  $\mathcal{N}_{U_i}$ . Then, the scalar reduction theory described here also applies as written to the trivial case  $r = l = 0$ .

Given any  $\Psi \in \mathcal{C}^\infty \underline{\mathcal{T}}^r \mathcal{N}$  and  $X \in \mathcal{X}_l^r(i)$ , then its full contraction  $\Psi(X)$  defines an element of  $\mathcal{C}^\infty \mathcal{N}_{U_i}$ . Moreover,  $\Psi|_{\mathcal{N}_{U_i}}$  can be entirely reconstructed from all of its contractions with the elements of  $\mathcal{X}_l^r(i)$ . Observe also that if  $\Phi \in \mathcal{C}^\infty \underline{\mathcal{T}}^r \mathcal{N}$  as well, then the orthonormality of the  $e_a^{(i)}$ 's and  $e_b^{(i)}$ 's implies that

$$(5.2) \quad \langle \Psi, \Phi \rangle|_{U_i} = \sum_{X \in \mathcal{X}_l^r(i)} [\Psi(X) \cdot \Phi(X)].$$

In particular, whenever  $\Psi = \Phi$ , we have

$$|\Psi|^2|_{U_i} = \sum_{X \in \mathcal{X}_l^r(i)} [\Psi(X)]^2.$$

We will use the collections  $\mathcal{X}_l^r(i)$  to systematically decompose tensorial quantities into a collection of scalar quantities. Under some conditions, this can be used to reduce difficult tensorial estimates to much easier scalar estimates. The primary aim of this section, then, is to show that this process reduces the main bilinear product estimates to their Euclidean counterparts. The fundamental observation here is the improved estimates of Proposition 4.9, which in particular is applicable to our motivating problem of null cones with bounded curvature flux.

We can also define an equivariant frame basis associated with the **(I)** condition. Indeed, given any  $1 \leq i \leq \mathbf{N}$ , we simply define the family

$$\mathcal{Z}(i) = \{\epsilon \partial_a^{(i)} \in \mathcal{C}^\infty \underline{\mathcal{T}}_0^1 \mathcal{N}_{U_i} \mid a \in \{1, 2\}\},$$

i.e., the equivariant transports of the  $\varphi_i$ -coordinate vector fields.<sup>40</sup> We will need this construction briefly for our final  $L_{x,t}^{\infty,2}$ -type estimate.

Next, we derive propagation estimates for our frame bases – elements of the  $\mathcal{X}_l^r(i)$ 's and  $\mathcal{Z}(i)$ 's – similar to those in Proposition 4.10. Again, these depend on the frame estimates in Propositions 4.7 and 4.9.

**Proposition 5.1.** Assume (4.2) and **(I)**, and fix  $1 \leq i \leq \mathbf{N}$ .

- If (4.4) also holds, then for any  $X \in \mathcal{X}_l^r(i)$ , we have

$$(5.3) \quad \|\nabla X\|_{L_{x,t}^{e,\infty}} \lesssim^c (r+l)(\mathbf{R}+\mathbf{d}).$$

- For any  $Z \in \mathcal{Z}(i)$ , we have

$$(5.4) \quad \|\nabla Z\|_{L_{x,t}^{2,\infty}} \lesssim^c \mathbf{R} + \|\nabla k\|_{L_{x,t}^{2,1}}.$$

<sup>40</sup>In fact, in [13, 22], analogous scalar reductions were made with respect to this family of frames. These coordinate fields, however, lacked the regularity required to fully reduce the desired bilinear product estimates. This necessitated the use of the geometric L-P theory of [12].

*Proof.* First, by (5.1) and the **(I)** condition,

$$\|X[0]\|_{L_x^\infty} \leq 1, \quad \|Z[0]\|_{L_x^\infty} \lesssim 1.$$

Also, by (4.6), we know that  $\mathcal{A}_v \simeq^c 1$ , where  $\mathcal{A}_v$  is the area  $(\mathcal{S}, \gamma[v])$ . As a result, we have from (5.1) and the **(I)** condition that

$$\|\nabla X[0]\|_{L_x^e} \leq (r+l)R, \quad \|\nabla Z[0]\|_{L_x^2} \leq R.$$

For (5.3), we simply apply (4.13) and recall (4.4):

$$\|\nabla X\|_{L_{x,t}^{e,\infty}} \lesssim^c \|\nabla X[0]\|_{L_x^e} + (r+l)d\|X[0]\|_{L_x^\infty} \leq (r+l)(R+d).$$

Similarly, (5.4) is obtained from (4.11):

$$\|\nabla Z\|_{L_{x,t}^{2,\infty}} \lesssim^c \|\nabla Z[0]\|_{L_x^2} + \|\nabla k\|_{L_{x,t}^{2,1}} \|Z[0]\|_{L_x^\infty} \lesssim R + \|\nabla k\|_{L_{x,t}^{2,1}}. \quad \square$$

Finally, given  $d > 0$  and non-negative integers  $r, l$ , we define the constant

$$(5.5) \quad C_l^r(d) = 2^{r+l}[1 + (r+l)(1+d)].$$

This is purely for convenience, as this constant will be present in many inequalities.

**5.2. Derivative Estimates.** We now prove some basic properties for our scalar reduction scheme, based on the assumptions in Section 4. First is the following comparison property for first derivative norms.

**Proposition 5.2.** *Let  $\Psi \in \mathcal{C}^\infty \underline{T}_1^r \mathcal{N}$ , and assume **(I)**.*

- *For each  $1 \leq i \leq N$ , the following identity holds on  $U_i$ :*

$$(5.6) \quad |\nabla_t \Psi|^2 + |\Psi|^2 = \sum_{X \in \mathcal{X}_l^r(i)} \{|\nabla_t[\Psi(X)]|^2 + |\Psi(X)|^2\}.$$

- *If (4.2) and (4.4) hold, then for each  $v \in [0, \delta)$  and  $1 \leq i \leq N$ ,*

$$(5.7) \quad \sum_{X \in \mathcal{X}_l^r(i)} \|\nabla[\Psi(X)][v]\|_{L_x^2} \lesssim_{N,R}^c C_l^r(d) (\|\nabla \Psi[v]\|_{L_x^2} + \|\Psi[v]\|_{L_x^2}).$$

*Proof.* First, (5.6) follows from (5.2) and the fact that each  $X \in \mathcal{X}_l^r(i)$  is  $t$ -parallel. For (5.7), we fix  $X \in \mathcal{X}_l^r(i)$ , and we define  $\nabla \Psi(X)$  and  $\Psi(\nabla X)$  to be the horizontal 1-forms mapping any  $Y \in \mathcal{C}^\infty \underline{T}_0^1 \mathcal{N}_{U_i}$  to  $(\nabla_Y \Psi)(X)$  and  $\Psi(\nabla_Y X)$ , respectively. Applying the Leibniz rule along with Hölder's inequality yields

$$\begin{aligned} \|\nabla[\Psi(X)][v]\|_{L_x^2} &\leq \|\nabla \Psi(X)[v]\|_{L_x^2} + \|\Psi(\nabla X)[v]\|_{L_x^2} \\ &\leq \|\nabla \Psi(X)[v]\|_{L_x^2} + \|\Psi[v]\|_{L_x^{e'}} \|\nabla X\|_{L_{x,t}^{e,\infty}}, \end{aligned}$$

where  $e^{-1} + (e')^{-1} = 2^{-1}$ . By (4.26) and (5.3),

$$\|\Psi[v]\|_{L_x^{e'}} \|\nabla X\|_{L_{x,t}^{e,\infty}} \lesssim_{N,R}^c (\|\nabla \Psi[v]\|_{L_x^2} + \|\Psi[v]\|_{L_x^2})(r+l)(1+d).$$

Squaring and then summing the above over all  $X \in \mathcal{X}_l^r(i)$  yields

$$\begin{aligned} \sum_{X \in \mathcal{X}_l^r(i)} \|\nabla[\Psi(X)][v]\|_{L_x^2}^2 &\lesssim \sum_{X \in \mathcal{X}_l^r(i)} [\|\nabla \Psi(X)[v]\|_{L_x^2}^2 + \|\Psi[v]\|_{L_x^{e'}}^2 \|\nabla X\|_{L_{x,t}^{e,\infty}}^2] \\ &\lesssim_{N,R}^c 2^{r+l}[1 + (r+l)^2(1+d)^2](\|\nabla \Psi[v]\|_{L_x^2}^2 + \|\Psi[v]\|_{L_x^2}^2), \end{aligned}$$

where we have recalled the identity (5.2). Finally, the desired inequality (5.7) follows from the above and from a trivial application of Hölder's inequality.  $\square$

**Corollary 5.3.** *Let  $\Psi \in \mathcal{C}^\infty \underline{\mathcal{T}}_d^r \mathcal{N}$ , and assume that **(I)**, (4.2), and (4.4) are satisfied. Then, for each  $1 \leq i \leq \mathbf{N}$ , we have the bound*

$$(5.8) \quad \sum_{X \in \mathcal{X}_l^r(i)} \|\Psi(X)\|_{H_{t,x}^1} \lesssim_{\mathbf{N}, \mathbf{R}}^c C_l^r(\mathbf{d}) \|\Psi\|_{H_{t,x}^1}.$$

*Proof.* This follows easily from (5.6) and (5.7).  $\square$

As a preliminary application of our scalar reduction, we prove a tensorial analogue to the  $L^2$ - $L^1$  Sobolev estimate (4.24). Note that such an estimate was not available using techniques based only on Proposition 4.13.

**Proposition 5.4.** *Assume **(I)**, and suppose both (4.2) and (4.4) hold. Then, for any  $\Psi \in \mathcal{C}^\infty \underline{\mathcal{T}}_d^r \mathcal{N}$  and  $v \in [0, \delta)$ , we have the estimate*

$$(5.9) \quad \|\Psi[v]\|_{L_x^2} \lesssim_{\mathbf{N}, \mathbf{R}, r+l, \mathbf{d}}^c \|\nabla \Psi[v]\|_{L_x^1} + \|\Psi[v]\|_{L_x^1}.$$

*Proof.* We begin by applying the scalar decomposition

$$\|\Psi[v]\|_{L_x^2}^2 \lesssim \sum_{i=1}^{\mathbf{N}} \sum_{X \in \mathcal{X}_l^r(i)} \|\Psi(X)[v]\|_{L_x^2}^2.$$

For each term on the right-hand side, we can apply (4.24):

$$\begin{aligned} \|\Psi[v]\|_{L_x^2}^2 &\lesssim_{\mathbf{N}, \mathbf{R}}^c \sum_{i=1}^{\mathbf{N}} \sum_{X \in \mathcal{X}_l^r(i)} [\|\nabla \Psi(X)[v]\|_{L_x^1}^2 + \|\Psi(X)[v]\|_{L_x^1}^2 + \|\Psi[v]\|_{L_x^{\mathbf{e}'}}^2 \|\nabla X\|_{L_{x,t}^{\mathbf{e}, \infty}}^2] \\ &= I_1 + I_2 + I_3, \end{aligned}$$

where  $\mathbf{e}^{-1} + (\mathbf{e}')^{-1} = 1$ . By trivial bounds for  $X$ , we have

$$I_1 + I_2 \lesssim_{\mathbf{N}} 2^{r+l} (\|\nabla \Psi[v]\|_{L_x^1}^2 + \|\Psi[v]\|_{L_x^1}^2).$$

For  $I_3$ , we apply (5.3):

$$I_3 \lesssim_{\mathbf{N}, \mathbf{R}}^c \|\Psi[v]\|_{L_x^{\mathbf{e}'}}^2 2^{r+l} (r+l)^2 (1+\mathbf{d})^2 \lesssim_{r+l, \mathbf{d}} \|\Psi[v]\|_{L_x^{\mathbf{e}'}}^2.$$

Moreover, since  $\mathbf{e}' < 2$ , then by the interpolation and Young inequalities,

$$I_3 \leq \varepsilon \|\Psi[v]\|_{L_x^2}^2 + C \|\Psi[v]\|_{L_x^1}^2,$$

where  $\varepsilon \ll 1$ , and where  $C$  is a large constant depending on  $\mathbf{N}$ ,  $\mathbf{R}$ ,  $\mathbf{c}$ ,  $r+l$ , and  $\mathbf{d}$ . Absorbing the  $\varepsilon$ -term in  $I_3$  into the left-hand side completes the proof.  $\square$

**5.3. Localized Besov Spaces.** Our next task is to define analogues on  $\mathcal{N}$  of the Besov-type  $B_x$ - and  $B_{t,x}$ -norms on  $[0, \delta) \times \mathbb{R}^2$ . Recall that from the **(I)** condition and the preceding discussion, we have devised a method of decomposing a horizontal tensor field  $\Psi \in \mathcal{C}^\infty \underline{\mathcal{T}}_d^r \mathcal{N}$  into localized scalar components

$$\Psi(X) \in \mathcal{C}^\infty \mathcal{N}_{U_i}, \quad X \in \mathcal{X}_l^r(i), \quad 1 \leq i \leq \mathbf{N},$$

which together carry the same information as  $\Psi$  itself. Furthermore, we have expressed  $\mathcal{S}$  as a union of bounded coordinate systems  $(U_i, \varphi_i)$ ,  $1 \leq i \leq \mathbf{N}$ . Therefore, we can define Besov-type norms for  $\Psi$  by porting each component  $\Psi(X)$  to  $\mathbb{R}^2$  via the  $\varphi_i$ 's, and then applying the Euclidean Besov norms to the resulting maps.

We assume once again the **(I)** condition, with the associated objects described in Section 4.3. For notational convenience, we will denote the equivariant transports

$\mathfrak{e}\eta_i$ ,  $\mathfrak{e}\tilde{\eta}_i$ ,  $\mathfrak{e}\vartheta_i$ , and  $\mathfrak{e}\varphi_i$  simply as  $\eta_i$ ,  $\tilde{\eta}_i$ ,  $\vartheta_i$ , and  $\varphi_i$ , respectively.<sup>41</sup> Moreover, throughout this subsection, we fix  $1 \leq i \leq N$ ,  $p \in [1, \infty]$ , and  $s \in [0, \infty)$ .

We begin by considering scalars  $f \in C^\infty \mathcal{S}$  and  $\phi \in C^\infty \mathcal{N}$ . Since  $\eta_i f$  is supported within  $U_i$ , then  $(\eta_i f) \circ \varphi_i^{-1}$  can be treated as a smooth function on  $\mathbb{R}^2$ . Similarly, recalling our notational convention  $\mathfrak{e}\eta_i \approx \eta_i$ , then  $(\eta_i \phi) \circ \varphi_i^{-1}$  can be treated as a smooth function on the Euclidean foliation  $[0, \delta) \times \mathbb{R}^2$ . As a result, we can construct coordinate analogues of the Euclidean Besov norms, defined in Section 3, by

$$\|f\|_{\mathcal{B}_x^s} = \sum_{i=1}^N \|(\eta_i f) \circ \varphi_i^{-1}\|_{B_x^s}, \quad \|\phi\|_{\mathcal{B}_{t,x}^{p,s}} = \sum_{i=1}^N \|(\eta_i \phi) \circ \varphi_i^{-1}\|_{B_{t,x}^{p,s}},$$

**Remark.** Note that these  $\mathcal{B}_x$ - and  $\mathcal{B}_{t,x}$ -norms depend very heavily on the various objects associated with the **(I)** condition. On the other hand, estimates involving these norms will depend only on the regularity constants  $N$  and  $R$ .

For tensorial cases, we simply add the scalar reduction process. More specifically, if  $F \in C^\infty T_l^r \mathcal{S}$  and  $\Psi \in C^\infty \underline{T}_l^r \mathcal{N}$ , then we can define the Besov-type norms

$$\begin{aligned} \|F\|_{\mathcal{B}_x^s(\gamma[v])} &= \sum_{1 \leq i \leq N} \sum_{X \in \mathcal{X}_l^r(i)} \|[\eta_i \cdot F(X[v])] \circ \varphi_i^{-1}\|_{B_x^s}, \\ \|\Psi\|_{\mathcal{B}_{t,x}^{p,s}} &= \sum_{1 \leq i \leq N} \sum_{X \in \mathcal{X}_l^r(i)} \|[\eta_i \cdot \Psi(X)] \circ \varphi_i^{-1}\|_{B_{t,x}^{p,s}}. \end{aligned}$$

For future reference, we note the following comparisons.

**Proposition 5.5.** Assume (4.1) and **(I)** hold, let  $p, q \in [1, \infty]$ , and let  $v \in [0, \delta)$ . If  $f \in C^\infty U_i$  and  $\phi \in C^\infty \mathcal{N}_{U_i}$ , then we have the comparisons

$$(5.10) \quad \|f\|_{L_x^q(\gamma[v])} \simeq^c \|f \circ \varphi_i^{-1}\|_{L_x^q}, \quad \|\phi\|_{L_{t,x}^{p,q}} \simeq^c \|\phi \circ \varphi_i^{-1}\|_{L_{t,x}^{p,q}}.$$

*Proof.* These are trivial consequences of (4.5), (4.21), and the **(I)** condition.  $\square$

To avoid confusion, in future proofs, we will let  $\partial$  denote the Euclidean *spatial* gradient on  $\mathbb{R}^2$  and  $[0, \delta) \times \mathbb{R}^2$ , and we will let  $\nabla$  denote the  $\gamma$ -connection on  $\mathcal{N}$ .

The following preliminary Besov estimate will be of importance later:<sup>42</sup>

**Lemma 5.6.** Assume (4.2) and **(I)**. If  $\phi \in C^\infty \mathcal{N}$ , then

$$(5.11) \quad \begin{aligned} \|\phi\|_{L_{t,x}^{\infty,\infty}} &\lesssim_{N,R}^c \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} [\|(\eta_i \cdot Z\phi) \circ \varphi_i^{-1}\|_{B_{t,x}^{\infty,0}} + \|(\tilde{\eta}_i \cdot Z\phi) \circ \varphi_i^{-1}\|_{L_{t,x}^{\infty,2}}] \\ &\quad + \|\phi\|_{L_{t,x}^{\infty,2}}. \end{aligned}$$

*Proof.* We begin by applying the classical sharp Besov embedding (3.5):

$$\begin{aligned} \|\phi\|_{L_{t,x}^{\infty,\infty}} &\lesssim^c \sum_{i=1}^N \|\eta_i \phi \circ \varphi_i^{-1}\|_{L_{t,x}^{\infty,\infty}} \\ &\lesssim \sum_{i=1}^N [\|\partial(\eta_i \phi \circ \varphi_i^{-1})\|_{B_{t,x}^{\infty,0}} + \|\eta_i \phi \circ \varphi_i^{-1}\|_{L_{t,x}^{\infty,2}}] \end{aligned}$$

<sup>41</sup>Here,  $\mathfrak{e}\varphi_i \approx \varphi_i$  denotes the  $\mathbb{R}^2$ -valued function, whose  $a$ -th component is defined to be the equivariant transport of the  $a$ -th component of  $\varphi_i$ .

<sup>42</sup>See Theorem 5.9.



$$\begin{aligned}
&\lesssim_N \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} \|(\eta_i \cdot Z\phi) \circ \varphi_i^{-1}\|_{B_{t,x}^{\infty,0}} + \|\phi\|_{L_{t,x}^{\infty,2}} \\
&\quad + \sum_{i=1}^N \|\partial(\eta_i \circ \varphi_i^{-1}) \cdot (\phi \circ \varphi_i^{-1})\|_{B_{t,x}^{\infty,0}}
\end{aligned}$$

Recall that the elements of  $\mathcal{Z}(i)$  correspond to the  $\varphi_i$ -coordinate derivatives. Now, it remains only to control the last term on the right-hand side.

Denote this last term by  $I$ . From (4.17), we have that

$$\|\partial(\eta_i \circ \varphi_i^{-1})\|_{L_{t,x}^{\infty,\infty}} \leq R, \quad \|\partial^2(\eta_i \circ \varphi_i^{-1})\|_{L_{t,x}^{\infty,\infty}} \leq R.$$

Thus, by a trivial embedding along with Hölder's inequality, then

$$\begin{aligned}
I &\lesssim \sum_{i=1}^N \{ \|\partial(\eta_i \circ \varphi_i^{-1}) \cdot (\phi \circ \varphi_i^{-1})\|_{L_{t,x}^{\infty,2}} + \|\partial(\eta_i \circ \varphi_i^{-1}) \cdot (\phi \circ \varphi_i^{-1})\|_{L_{t,x}^{\infty,2}} \} \\
&\lesssim_R \sum_{i=1}^N [ \|\phi \circ \varphi_i^{-1}\|_{L_{t,x}^{\infty,2}} + \|(\tilde{\eta}_i \circ \varphi_i^{-1}) \cdot \partial(\phi \circ \varphi_i^{-1})\|_{L_{t,x}^{\infty,2}} ] \\
&= I_1 + I_2.
\end{aligned}$$

Note that we could freely insert the factor  $\tilde{\eta}_i$  in the above manipulation, since  $\tilde{\eta}_i$  is identically 1 on the support of  $\eta_i$ . The estimate (5.11) follows, since by definition,

$$I_1 \lesssim_N \|\phi\|_{L_{t,x}^{\infty,2}}, \quad I_2 \lesssim \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} \|(\tilde{\eta}_i Z\phi) \circ \varphi_i^{-1}\|_{L_{t,x}^{\infty,2}}. \quad \square$$

**5.4. Non-Integrated Product Estimates.** We are now ready to prove the main estimates of this paper. We begin here with the geometric tensorial analogues of the nonintegrated bilinear product estimates. These proofs demonstrate the general methodology of applying the scalar reduction formalism constructed above in order to port Euclidean space arguments to the evolving geometric setting.

**Theorem 5.7.** *Assume (4.2), (4.4), and (I), and let*

$$F \in \mathcal{C}^\infty T_{l_1}^{r_1} \mathcal{S}, \quad G \in \mathcal{C}^\infty T_{l_2}^{r_2} \mathcal{S}, \quad \Psi \in \mathcal{C}^\infty \underline{T}_{l_1}^{r_1} \mathcal{N}, \quad \Phi \in \mathcal{C}^\infty \underline{T}_{l_2}^{r_2} \mathcal{N}.$$

- If  $v \in [0, \delta)$  and  $p \in [1, \infty]$ , then

$$\begin{aligned}
(5.12) \quad &\|F \otimes G\|_{\mathcal{B}_x^0(\gamma[v])} \lesssim_{N,R} C_{l_1}^{r_1}(d) (\|\nabla F\|_{L_x^2(\gamma[v])} + \|F\|_{L_x^\infty(\gamma[v])}) \|G\|_{\mathcal{B}_x^0(\gamma[v])}, \\
&\|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{p,0}} \lesssim_{N,R} C_{l_1}^{r_1}(d) (\|\nabla \Phi\|_{L_{t,x}^{\infty,2}} + \|\Phi\|_{L_{t,x}^{\infty,\infty}}) \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}}.
\end{aligned}$$

- If  $\Psi$  is  $t$ -parallel, then

$$(5.13) \quad \|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{2,0}} \lesssim_{N,R} C_{l_1}^{r_1}(d) (\|\nabla \Phi\|_{L_{t,x}^{2,2}} + \|\Phi\|_{L_{x,t}^{\infty,2}}) \|\Psi[0]\|_{\mathcal{B}_x^0}.$$

- In addition, the following “improved” estimate holds:

$$(5.14) \quad \delta \|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{\infty,0}} \lesssim_{N,R} C_{l_1}^{r_1}(d) C_{l_2}^{r_2}(d) \|\Phi\|_{H_{t,x}^1} \|\Psi\|_{H_{t,x}^1}.$$

*Proof.* For (5.12), first we apply the parallel scalar reduction:

$$\|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{p,0}} = \sum_{i=1}^N \sum_{W \in \mathcal{X}_{l_1+l_2}^{r_1+r_2}(i)} \|[\eta_i \cdot (\Phi \otimes \Psi)(W)] \circ \varphi_i^{-1}\|_{B_{t,x}^{p,0}}$$

$$= \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|[\tilde{\eta}_i \Phi(X) \cdot \eta_i \Psi(Y)] \circ \varphi_i^{-1}\|_{B_{t,x}^{p,0}}.$$

Observe that the elements of  $\mathcal{X}_{l_1+l_2}^{r_1+r_2}(i)$  are precisely the tensor products of elements of  $\mathcal{X}_{l_1}^{r_1}(i)$  and  $\mathcal{X}_{l_2}^{r_2}(i)$ . Applying the Euclidean analogue (3.12) of our desired estimate along with the **(I)** condition and (4.20), then we obtain <sup>43</sup>

$$\begin{aligned} \|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{p,0}} &\lesssim \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \{\|\partial[\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}]\|_{L_{t,x}^{\infty,2}} + \|\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}\|_{L_{t,x}^{\infty,\infty}}\} \\ &\quad \cdot \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|\eta_i \Psi(Y) \circ \varphi_i^{-1}\|_{B_{t,x}^{p,0}} \\ &\lesssim_{\mathbf{N},\mathbf{R}}^c \sup_{1 \leq i \leq N} \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \{\|\nabla[\Phi(X)]\|_{L_{t,x}^{\infty,2}} + \|\Phi(X)\|_{L_{t,x}^{\infty,\infty}}\} \cdot \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}}, \end{aligned}$$

where in the last step, we applied the definition of the  $\mathcal{B}_{t,x}$ -norm. It remains only to reconstruct  $\Phi$  and  $\Psi$  from their scalar components. For this, we apply (5.7):

$$\|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{p,0}} \lesssim_{\mathbf{N},\mathbf{R}}^c C_{l_1}^{r_1}(\mathbf{d})(\|\nabla \Phi\|_{L_{t,x}^{\infty,2}} + \|\Phi\|_{L_{t,x}^{\infty,\infty}}) \cdot \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}}.$$

This completes the proof of the second part of (5.12); the first inequality in (5.12) is proved analogously, using the corresponding inequality in (3.12).

For (5.13), we employ an analogous scalar decomposition:

$$\|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{2,0}} = \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|[\tilde{\eta}_i \Phi(X) \cdot \eta_i \Psi(Y)] \circ \varphi_i^{-1}\|_{B_{t,x}^{2,0}}.$$

Since  $\Psi$  is  $t$ -parallel, then  $\eta_i \Psi(Y)$  is  $t$ -parallel for each  $1 \leq i \leq N$  and  $Y \in \mathcal{X}_{l_2}^{r_2}(i)$ . Thus, we can apply (3.13) to the above right-hand side.

$$\begin{aligned} \|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{2,0}} &\lesssim_{\mathbf{N},\mathbf{R}}^c \sup_{1 \leq i \leq N} \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \{\|\nabla[\Phi(X)]\|_{L_{t,x}^{2,2}} + \|\Phi(X)\|_{L_{x,t}^{\infty,2}}\} \\ &\quad \cdot \sum_{i=1}^N \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|\eta_i \Psi(Y) \circ \varphi_i^{-1}[0]\|_{B_x^0}. \end{aligned}$$

Finally, by (5.7) and the definition of the  $\mathcal{B}_x^0$ -norm, we obtain

$$\|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{2,0}} \lesssim_{\mathbf{N},\mathbf{R}}^c C_{l_1}^{r_1}(\mathbf{d})(\|\nabla \Phi\|_{L_{t,x}^{2,2}} + \|\Phi\|_{L_{x,t}^{2,\infty}}) \|\Psi[0]\|_{\mathcal{B}_x^0}.$$

To prove (5.14), we again decompose

$$\delta \|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{\infty,0}} = \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \delta \|[\tilde{\eta}_i \Phi(X) \cdot \eta_i \Psi(Y)] \circ \varphi_i^{-1}\|_{B_{t,x}^{\infty,0}}.$$

Applying (3.20), then

$$\delta \|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{\infty,0}} \lesssim \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \|\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}\|_{H_{t,x}^1} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|\eta_i \Psi(Y) \circ \varphi_i^{-1}\|_{H_{t,x}^1}.$$

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<sup>43</sup>Again, recall  $\partial$  denotes Euclidean spatial derivatives, while  $\nabla$  denotes  $\gamma$ -derivatives on  $\mathcal{N}$ .

Similar to the preceding proofs, we can now apply (4.20), the **(I)** condition, and (5.8) in order to obtain the desired estimate (5.14).  $\square$

**Remark.** Due to the nature of our scalar reductions, the estimates (5.12)-(5.14) still hold if tensor products  $\Phi \otimes \Psi$  and  $F \otimes G$  on the left-hand sides are replaced instead by zero or more of the following operations applied to those products:

- Contractions.
- Metric contractions.
- Volume form contractions.

**5.5. Integrated Product Estimates.** We move on to the integrated product estimates. The basic strategy is the same as before, except that we require one additional observation: contractions by  $t$ -parallel fields commute with the covariant integrals  $\mathfrak{T}$ . This is due to the fact that such contractions commute with  $\nabla_t$ .

**Theorem 5.8.** Let  $\Psi \in \mathcal{C}^\infty \underline{T}_1^{r_1} \mathcal{N}$  and  $\Phi \in \mathcal{C}^\infty \underline{T}_2^{r_2} \mathcal{N}$ , and assume (4.2), (4.4), and **(I)**. Then, the following integrated bilinear product estimates hold:

$$(5.15) \quad \|\mathfrak{T}(\Phi \otimes \Psi)\|_{\mathcal{B}_{t,x}^{\infty,0}} \lesssim_{\mathbf{N},\mathbf{R}}^c C_{l_1}^{r_1}(\mathbf{d})(\|\nabla \Phi\|_{L_{t,x}^{2,2}} + \|\Phi\|_{L_{x,t}^{\infty,2}}) \|\Psi\|_{\mathcal{B}_{t,x}^{2,0}},$$

$$(5.16) \quad \|\Phi \otimes \mathfrak{T}\Psi\|_{\mathcal{B}_{t,x}^{2,0}} \lesssim_{\mathbf{N},\mathbf{R}}^c C_{l_1}^{r_1}(\mathbf{d})(\|\nabla \Phi\|_{L_{t,x}^{2,2}} + \|\Phi\|_{L_{x,t}^{\infty,2}}) \|\Psi\|_{\mathcal{B}_{t,x}^{1,0}},$$

$$(5.17) \quad \delta \|\mathfrak{T}(\nabla_t \Phi \otimes \Psi)\|_{\mathcal{B}_{t,x}^0} \lesssim_{\mathbf{N},\mathbf{R}}^c C_{l_1}^{r_1}(\mathbf{d}) C_{l_2}^{r_2}(\mathbf{d}) \|\Phi\|_{H_{t,x}^1} \|\Psi\|_{H_{t,x}^1}.$$

*Proof.* For (5.15), we once again resort to our usual scalar reduction:

$$\|\mathfrak{T}(\Phi \otimes \Psi)\|_{\mathcal{B}_{t,x}^{\infty,0}} = \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|\mathfrak{T}[\tilde{\eta}_i \Phi(X) \cdot \eta_i \Psi(Y)] \circ \varphi_i^{-1}\|_{\mathcal{B}_{t,x}^{\infty,0}}.$$

Here, we have noted that  $\mathfrak{T}$  commutes with contractions with both  $X$  and  $Y$ , as well as with multiplication by the ( $t$ -independent) cutoff functions  $\eta_i$  and  $\tilde{\eta}_i$ .

Applying (3.16) to the above, we have

$$\begin{aligned} \|\mathfrak{T}(\Phi \otimes \Psi)\|_{\mathcal{B}_{t,x}^0} &\lesssim \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \{ \|\partial[\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}]\|_{L_{t,x}^{2,2}} + \|\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}\|_{L_{x,t}^{\infty,2}} \} \\ &\quad \cdot \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|\eta_i \Psi(Y) \circ \varphi_i^{-1}\|_{\mathcal{B}_{t,x}^{2,0}}. \end{aligned}$$

As in the preceding proofs, recalling (4.20), the **(I)** condition, (5.7), and the definition of the  $\mathcal{B}_{t,x}$ -norms, then the above becomes

$$\|\mathfrak{T}(\Phi \otimes \Psi)\|_{\mathcal{B}_{t,x}^{\infty,0}} \lesssim_{\mathbf{N},\mathbf{R}}^c C_{l_1}^{r_1}(\mathbf{d})(\|\nabla \Phi\|_{L_{t,x}^{2,2}} + \|\Phi\|_{L_{x,t}^{\infty,2}}) \|\Psi\|_{\mathcal{B}_{t,x}^{2,0}},$$

which completes the proof of (5.15).

Next, for (5.16), we once again apply the scalar reduction argument, which yields

$$\|\Phi \otimes \mathfrak{T}\Psi\|_{\mathcal{B}_{t,x}^{2,0}} = \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|\{\tilde{\eta}_i \Phi(X) \cdot \mathfrak{T}[\eta_i \Psi(Y)]\} \circ \varphi_i^{-1}\|_{\mathcal{B}_{t,x}^{2,0}}.$$

Again,  $\mathfrak{T}$  commutes with contractions with  $Y$  and  $\eta_i$ . Applying (3.17) yields

$$\|\Phi \otimes \mathfrak{T}\Psi\|_{\mathcal{B}_{t,x}^{2,0}} \lesssim \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \{ \|\partial[\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}]\|_{L_{t,x}^{2,2}} + \|\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}\|_{L_{x,t}^{\infty,2}} \}$$

$$\cdot \sum_{Y \in \mathcal{X}_{l_2}^{r_2}} \|\eta_i \Psi(Y) \circ \varphi_i^{-1}\|_{B_{t,x}^{1,0}}.$$

From here, the proof is completed using the same reasoning as for (5.15).

Finally, for (5.17), the usual scalar reduction yields

$$\delta \|\mathfrak{T}(\nabla_t \Phi \otimes \Psi)\|_{\mathcal{B}_{t,x}^{\infty,0}} = \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \delta \|\mathfrak{T}\{\nabla_t[\tilde{\eta}_i \Phi(X)]\eta_i \Psi(Y)\} \circ \varphi_i^{-1}\|_{B_{t,x}^{\infty,0}}.$$

On the right-hand side, since the quantities being differentiated and integrated are all scalar, then  $\nabla_t$  and  $\mathfrak{T}$  are the standard derivative and integral, respectively, with respect to the variable  $t$ . Thus, we can apply (3.21) to the above:

$$\delta \|\mathfrak{T}(\nabla_t \Phi \otimes \Psi)\|_{\mathcal{B}_{t,x}^{\infty,0}} \lesssim \sum_{i=1}^N \sum_{X \in \mathcal{X}_{l_1}^{r_1}(i)} \|\tilde{\eta}_i \Phi(X) \circ \varphi_i^{-1}\|_{H_{t,x}^1} \sum_{Y \in \mathcal{X}_{l_2}^{r_2}(i)} \|\eta_i \Psi(Y) \circ \varphi_i^{-1}\|_{H_{t,x}^1}.$$

Like before, by (4.20), the **(I)** condition, and (5.8), we obtain (5.17).  $\square$

**Remark.** Again, the estimates (5.15)-(5.17) still hold if  $\Phi \otimes \Psi$  on the respective left-hand sides are replaced instead by zero or more contractions, metric contractions, and volume form contractions applied to  $\Phi \otimes \Psi$ .

**5.6. A Sharp Trace Estimate.** In [11, 13, 22], one controlled the  $L_{x,t}^{\infty,2}$ -norm via a “sharp trace” estimate, which was obtained using bilinear product estimates analogous to those in Theorem 5.8. We now accomplish this same objective in our current setting. The process is essentially the same as found in [13, 22], but once again, we will state and prove this result in terms of our localized Besov spaces.

**Theorem 5.9.** Assume (4.2), (4.4), and **(I)** hold. Let  $\Psi \in C^\infty \underline{T}^r \mathcal{N}$ , and suppose  $\Psi_1, \Psi_2 \in C^\infty \underline{T}_{l+1}^r \mathcal{N}$  are such that the decomposition

$$\nabla \Psi = \delta \nabla_t \Psi_1 + \Psi_2$$

holds. Then, we have the sharp trace estimate

$$(5.18) \quad \begin{aligned} \|\Psi\|_{L_{x,t}^{\infty,2}} &\lesssim_{\mathbf{N},\mathbf{R}}^c \|\Psi\|_{H_{t,x}^1} + [C_{l+1}^r(\mathbf{d})]^2 (1 + \delta^{\frac{1}{2}} \|k\|_{H_{t,x}^1}) \|\Psi_1\|_{H_{t,x}^1} \\ &\quad + C_{l+1}^r(\mathbf{d}) (1 + \delta^{\frac{1}{2}} \|k\|_{H_{t,x}^1}) \|\Psi_2\|_{\mathcal{B}_{t,x}^{2,0}}. \end{aligned}$$

*Proof.* Applying Lemma 5.6, we have the estimate

$$\begin{aligned} \|\Psi\|_{L_{x,t}^{\infty,2}}^2 &\lesssim \|\mathfrak{T}|\Psi|^2\|_{L_{t,x}^{\infty,\infty}} \\ &\lesssim_{\mathbf{N},\mathbf{R}}^c \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} [\|\mathfrak{T}(\eta_i Z |\Psi|^2) \circ \varphi_i^{-1}\|_{B_{t,x}^{\infty,0}} + \|\mathfrak{T}(\tilde{\eta}_i Z |\Psi|^2) \circ \varphi_i^{-1}\|_{L_{t,x}^{\infty,2}}] \\ &\quad + \|\mathfrak{T}|\Psi|^2\|_{L_{t,x}^{\infty,2}} \\ &= I_1 + I_2 + I_3. \end{aligned}$$

The lower-order term  $I_3$  can be handled using (4.8) and (4.26):

$$I_3 \lesssim^c \|\Psi\|_{L_{x,t}^{4,2}}^2 \lesssim \|\Psi\|_{L_{t,x}^{2,4}}^2 \lesssim_{\mathbf{N},\mathbf{R}}^c \|\Psi\|_{H_{t,x}^1}^2.$$

If  $1 \leq i \leq N$  and  $Z \in \mathcal{Z}(i)$ , then we can treat  $\tilde{\eta}_i Z$  as a global vector field, i.e., as an element of  $\mathcal{C}^\infty \underline{T}_0^1 \mathcal{N}$ . As a result, applying (4.8) and (4.10), we obtain

$$\begin{aligned} I_2 &\lesssim^c \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} \|\mathfrak{T} \langle \nabla \Psi, \tilde{\eta}_i Z \otimes \Psi \rangle\|_{L_{t,x}^{\infty,2}} \\ &\lesssim^c \|\nabla \Psi\|_{L_{t,x}^{2,2}} \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} \|\tilde{\eta}_i Z \otimes \Psi\|_{L_{x,t}^{\infty,2}} \\ &\lesssim_N^c \|\Psi\|_{H_{t,x}^1} \|\Psi\|_{L_{x,t}^{\infty,2}}. \end{aligned}$$

By a similar process, in conjunction with our decomposition for  $\nabla \Psi$ , we have

$$\begin{aligned} I_1 &\lesssim \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} \|\mathfrak{T} \langle \nabla \Psi, \tilde{\eta}_i Z \otimes \Psi \rangle\|_{\mathcal{B}_{t,x}^{\infty,0}} \\ &\lesssim \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} (\|\mathfrak{T} \langle \delta \nabla_t \Psi_1, \tilde{\eta}_i Z \otimes \Psi \rangle\|_{\mathcal{B}_{t,x}^{\infty,0}} + \|\mathfrak{T} \langle \Psi_2, \tilde{\eta}_i Z \otimes \Psi \rangle\|_{\mathcal{B}_{t,x}^{\infty,0}}). \end{aligned}$$

Applying (5.15) and (5.17), along with the remark following Theorem 5.8, then

$$\begin{aligned} I_1 &\lesssim_{N,R}^c [C_{l+1}^r(d)]^2 \|\Psi_1\|_{H_{t,x}^1} \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} \|\tilde{\eta}_i Z \otimes \Psi\|_{H_{t,x}^1} \\ &\quad + C_{l+1}^r(d) \|\Psi_2\|_{\mathcal{B}_{t,x}^{2,0}} \sum_{i=1}^N \sum_{Z \in \mathcal{Z}(i)} [\|\nabla(\tilde{\eta}_i Z \otimes \Psi)\|_{L_{t,x}^{2,2}} + \|\tilde{\eta}_i Z \otimes \Psi\|_{L_{x,t}^{\infty,2}}]. \end{aligned}$$

Fix now any such  $\tilde{\eta}_i Z$ . First, by (4.10), we have

$$\|\tilde{\eta}_i Z \otimes \Psi\|_{L_{x,t}^{\infty,2}} \lesssim^c \|\Psi\|_{L_{x,t}^{\infty,2}}.$$

Moreover, applying Hölder's inequality along with (4.10) and (4.20) yields

$$\|\tilde{\eta}_i Z \otimes \Psi\|_{H_{t,x}^1} \lesssim^c \|\Psi\|_{H_{t,x}^1} + (\delta \|\nabla_t Z\|_{L_{x,t}^{2,\infty}} + \|\nabla Z\|_{L_{x,t}^{2,\infty}}) \|\Psi\|_{L_{x,t}^{\infty,2}}.$$

Since  $|\nabla_t Z| \lesssim |k| |Z|$ , then by (4.29),

$$\delta \|\nabla_t Z\|_{L_{x,t}^{2,\infty}} \|\Psi\|_{L_{x,t}^{\infty,2}} \lesssim_{N,R}^c \delta^{\frac{1}{2}} \|k\|_{H_{t,x}^1} \|\Psi\|_{L_{x,t}^{\infty,2}}.$$

Moreover, applying (5.4), we obtain

$$\|\nabla Z\|_{L_{x,t}^{2,\infty}} \|\Psi\|_{L_{x,t}^{\infty,2}} \lesssim^c (1 + \delta^{\frac{1}{2}} \|k\|_{H_{t,x}^1}) \|\Psi\|_{L_{x,t}^{\infty,2}}.$$

Finally, combining the above, we obtain

$$\begin{aligned} \|\Psi\|_{L_{x,t}^{\infty,2}}^2 &\lesssim_{N,R} I_1 + I_2 + I_3 \\ &\lesssim_{N,R}^c \|\Psi\|_{H_{t,x}^1}^2 + \|\Psi\|_{H_{t,x}^1} \|\Psi\|_{L_{x,t}^{\infty,2}} \\ &\quad + \{[C_{l+1}^r(d)]^2 \|\Psi_1\|_{H_{t,x}^1} + C_{l+1}^r(d) \|\Psi_2\|_{\mathcal{B}_{t,x}^{2,0}}\} \|\Psi\|_{H_{t,x}^1} \\ &\quad + \{[C_{l+1}^r(d)]^2 \|\Psi_1\|_{H_{t,x}^1} + C_{l+1}^r(d) \|\Psi_2\|_{\mathcal{B}_{t,x}^{2,0}}\} (1 + \delta^{\frac{1}{2}} \|k\|_{H_{t,x}^1}) \|\Psi\|_{L_{x,t}^{\infty,2}}. \end{aligned}$$

An application of a weighted Cauchy inequality completes the proof.  $\square$

## 6. GEOMETRIC LITTLEWOOD-PALEY THEORY

In the preceding section, we discussed bilinear product estimates, in particular analogues of those found in [13], on general surface foliations with evolving geometries satisfying some weak regularity assumptions. In particular, we demonstrated that these estimates can in fact be derived almost immediately from their Euclidean counterparts via scalar reductions. Moreover, we showed how this general setting applies to our main problem: regular null cones with finite curvature flux. Now, one final task remains, which is to compare our coordinate-based Besov norms to analogous Besov-type norms constructed from a more geometric L-P theory.

In this section, we construct a fully geometric L-P theory from spectral decompositions of the (Böchner) Laplacian. More specifically, we adopt the same ideas as in [4], but also for tensorial quantities. Afterwards, we define the invariant analogues of the  $\mathcal{B}_x$ - and  $\mathcal{B}_{t,x}$ -norms from this theory, and we then obtain the desired comparisons between these two versions of Besov-type norms.

**Remark.** *An alternative approach is to use the geometric L-P theory of [12], based on the heat flow. In fact, one can attain Besov norm comparisons using either geometric L-P theory, with mostly the same proofs. Since the spectral version is much easier to rigorously construct and utilize, we opt for this route in this paper.*

**6.1. Spectral Decompositions.** For now, we restrict our attention to only a single instance of  $\mathcal{S}$ . We fix a Riemannian metric  $h$  on  $\mathcal{S}$ , and we assume all relevant objects on  $\mathcal{S}$  are defined with respect to  $h$  (e.g., tensor norms, covariant derivatives, integral norms). For technical purposes, we consider the Hilbert space

$$L^2 T_l^r \mathcal{S} = L_{(h)}^2 T_l^r \mathcal{S},$$

the completion of  $\mathcal{C}^\infty T_l^r \mathcal{S}$  with respect to the above  $L_x^2$ -norm. Consider now the negative Laplacian  $-\Delta = -\Delta^{(h)}$ , which we can interpret as an unbounded operator on  $L^2 T_l^r \mathcal{S}$ . Via standard methods,  $-\Delta$  can be extended into a positive self-adjoint operator, which then has a spectral decomposition <sup>44</sup>

$$-\Delta = \int_0^\infty \lambda \cdot dE_\lambda.$$

Let  $\varsigma$  and  $\varsigma_k$ , where  $k \in \mathbb{Z}$ , denote the family of cutoff functions defined in Section 3.1. We can interpret these now as functions on  $\mathbb{C}$ . The geometric Littlewood-Paley operators are defined on  $L^2 T_l^r \mathcal{S}$  as smoothed spectral projections of  $-\Delta$ :

$$P_k = \varsigma_{2k}(-\Delta), \quad P_- = \chi_{\{0\}}(-\Delta).$$

In particular,  $P_-$  is precisely the  $L^2$ -projection onto the kernel of  $\Delta$ . Like for the classical setting, given any  $k \in \mathbb{Z}$ , we can define the aggregated operators

$$P_{<k} = P_- + \sum_{l < k} P_l, \quad P_{\geq k} = \sum_{l \geq k} P_l,$$

where the summations are interpreted as pointwise, i.e., with respect to the strong operator topology. Also, for each  $k \in \mathbb{Z}$ , we have  $P_{<k} + P_{\geq k} = I$  (in the strong topology), where  $I$  denotes the identity operator on  $L^2 T_l^r \mathcal{S}$ .

Note that the  $P_k$ 's are almost pairwise orthogonal, that is,

$$(6.1) \quad P_k P_l \equiv 0, \quad k, l \in \mathbb{Z}, \quad |k - l| > 1.$$

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<sup>44</sup>This spectral decomposition is in fact discrete; see [7].

As a result of this, we can use the notations  $P_{\sim k}$ ,  $P_{\lesssim k}$ , and  $P_{\gtrsim k}$  in the same fashion as for the classical L-P operators in Section 3. Furthermore, another important analogue of the Euclidean theory is the identity

$$2^{2k}P_k \sim \Delta \tilde{P}_k, \quad k \in \mathbb{Z},$$

where  $\tilde{P}_k = \tilde{\zeta}_{2k}(-\Delta)$  is another smoothed spectral projection. Again, the  $\tilde{P}_k$ 's will have mostly the same estimates as the  $P_k$ 's. In summary, we have a similar bag of tricks for our geometric L-P theory as we did for our classical theory.

We now list some basic properties satisfied by our geometric L-P operators.

**Proposition 6.1.** *Let  $k \in \mathbb{Z}$ , and let  $\Psi \in C^\infty T_l^r \mathcal{S}$ .*

- $P_k$ ,  $P_{\geq k}$ , and  $P_{< k}$  are bounded operators, i.e.,

$$(6.2) \quad \|P_k \Psi\|_{L_x^2} + \|P_{< k} \Psi\|_{L_x^2} + \|P_{\geq k} \Psi\|_{L_x^2} \lesssim \|\Psi\|_{L_x^2}.$$

- The following “finite band” estimates hold:

$$(6.3) \quad \begin{aligned} \|\Delta P_k \Psi\|_{L_x^2} &\lesssim 2^{2k} \|\Psi\|_{L_x^2}, & \|P_k \Psi\|_{L_x^2} &\lesssim 2^{-2k} \|\Delta \Psi\|_{L_x^2}, \\ \|\Delta P_{< k} \Psi\|_{L_x^2} &\lesssim 2^{2k} \|\Psi\|_{L_x^2}, & \|P_{\geq k} \Psi\|_{L_x^2} &\lesssim 2^{-2k} \|\Delta \Psi\|_{L_x^2}. \end{aligned}$$

- The following “finite band” estimates hold:<sup>45</sup>

$$(6.4) \quad \begin{aligned} \|\nabla P_k \Psi\|_{L_x^2} &\lesssim 2^k \|\Psi\|_{L_x^2}, & \|P_k \nabla \Psi\|_{L_x^2} &\lesssim 2^k \|\Psi\|_{L_x^2}, \\ \|\nabla P_{< k} \Psi\|_{L_x^2} &\lesssim 2^k \|\Psi\|_{L_x^2}, & \|P_{< k} \nabla \Psi\|_{L_x^2} &\lesssim 2^k \|\Psi\|_{L_x^2}. \end{aligned}$$

- The following “reverse finite band” estimate holds:

$$(6.5) \quad \|P_k \Psi\|_{L_x^2} \lesssim 2^{-k} \|\nabla \Psi\|_{L_x^2}, \quad \|P_{\geq k} \Psi\|_{L_x^2} \lesssim 2^{-k} \|\nabla \Psi\|_{L_x^2}.$$

*Proof.* First, (6.2) and (6.3) are direct consequences of the definitions of the  $P_k$ 's and  $P_-$ . The first estimate in (6.4) follows from (6.3) via an integration by parts:

$$\|\nabla P_k \Psi\|_{L_x^2}^2 \leq \|(-\Delta) P_k \Psi\|_{L_x^2} \|P_k \Psi\|_{L_x^2} \lesssim 2^{2k} \|\Psi\|_{L_x^2}^2.$$

That  $P_k \nabla$  is similarly bounded follows from the previous bound by a standard duality argument. For the first part of (6.5), we can apply the above:

$$\|P_k \Psi\|_{L_x^2} = 2^{-2k} \|\tilde{P}_k(-\Delta) \Psi\|_{L_x^2} \lesssim 2^k \|\nabla \Psi\|_{L_x^2}.$$

Finally, the remaining estimates of (6.4) and (6.5) follow immediately from the already completed estimates by direct summations.  $\square$

In addition, if  $(\mathcal{S}, h)$  has the right first-order Sobolev estimates, then we obtain “weak Bernstein” estimates for our geometric L-P operators.

**Proposition 6.2.** *Fix  $v \in [0, \delta)$ , and suppose now  $h = \gamma[v]$ . Moreover, assume both (4.2) and (I). If  $F \in C^\infty T_l^r \mathcal{S}$ ,  $k \geq 0$ ,  $q \in (2, \infty)$ , and  $(q')^{-1} = 1 - q^{-1}$ , then*

$$(6.6) \quad \begin{aligned} \|P_k F\|_{L_x^q} &\lesssim_{\mathbb{N}, \mathbb{R}, q}^c 2^{(1-\frac{2}{q})k} \|F\|_{L_x^2}, & \|P_{< 0} F\|_{L_x^q} &\lesssim_{\mathbb{N}, \mathbb{R}, q}^c \|F\|_{L_x^2}, \\ \|P_k F\|_{L_x^2} &\lesssim_{\mathbb{N}, \mathbb{R}, q'}^c 2^{(\frac{2}{q'}-1)k} \|F\|_{L_x^{q'}}, & \|P_{< 0} F\|_{L_x^2} &\lesssim_{\mathbb{N}, \mathbb{R}, q'}^c \|F\|_{L_x^{q'}}. \end{aligned}$$

*Proof.* The first two inequalities are proved using (4.26) in conjunction with (6.4). The remaining two inequalities follow by duality.  $\square$

<sup>45</sup>In the expressions  $P_k \nabla \Psi$  and  $P_{< k} \nabla \Psi$ , the L-P operators are of course those on  $L^2 T_{l+1}^r \Psi$ .

Finally, we return to abstract foliation  $\mathcal{N}$ . On every  $(\mathcal{S}, \gamma[v])$ , we can construct this spectral L-P theory. We can then aggregate the operators  $P_k$ ,  $P_{<k}$ ,  $P_{\geq k}$  for any  $k \in \mathbb{Z}$ , so that they act on  $\mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , with respect to  $\gamma[v]$  on each  $\mathcal{S}_v$ .

Now, given any  $s \in [0, \infty)$ ,  $p \in [1, \infty]$ ,  $F \in \mathcal{C}^\infty T_l^r \mathcal{S}$ , and  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , we can naturally define the following *geometric, tensorial* Besov-type norms:

$$\begin{aligned} \|F\|_{\mathcal{B}_x^s(h)} &= \sum_{k \geq 0} 2^{sk} \|P_k F\|_{L_x^2(h)} + \|P_{<0} F\|_{L_x^2(h)}, \\ \|\Psi\|_{\mathcal{B}_{t,x}^{p,s}} &= \|\Psi\|_{\mathcal{B}_{t,x}^{p,s}(\gamma)} = \sum_{k \geq 0} 2^{sk} \|P_k \Psi\|_{L_{t,x}^{p,2}} + \|P_{<0} \Psi\|_{L_{t,x}^{p,2}}. \end{aligned}$$

In the first identity, the geometric L-P operators are with respect to  $h$ . These are the direct analogues of the coordinate-based  $\mathcal{B}_x$ - and  $\mathcal{B}_{t,x}$ -norms.<sup>46</sup> The main objective of the remainder of this section, then, is to show that with the relevant assumptions, the various  $\mathcal{B}$ - and  $\mathcal{B}$ -norms are comparable.

**6.2. Scalar Reduction Estimates.** One main technical step toward our desired Besov comparison result is to show that the scalar reduction process described in Section 5 is also applicable to our geometric Besov norms. The main technical component of this process is the following non-sharp variant of (5.12).

**Lemma 6.3.** *Assume (4.2) and (I), and let*

$$F \in \mathcal{C}^\infty T_{l_1}^{r_1} \mathcal{S}, \quad G \in \mathcal{C}^\infty T_{l_2}^{r_2} \mathcal{S}, \quad \Psi \in \mathcal{C}^\infty \underline{T}_{l_1}^{r_1} \mathcal{N}, \quad \Phi \in \mathcal{C}^\infty \underline{T}_{l_2}^{r_2} \mathcal{N}.$$

- For any  $v \in [0, \delta)$  and  $k, l \in \mathbb{Z}$ , with  $k \geq 0$ , then

$$\begin{aligned} (6.7) \quad \|P_k(F \otimes P_l G)\|_{L_x^2} &\lesssim_{\mathbf{N}, \mathbf{R}, \mathbf{e}} 2^{-|k-l|} (\|\nabla F\|_{L_x^e} + \|F\|_{L_x^\infty}) \|P_{\sim l} G\|_{L_x^2}, \\ \|P_k(F \otimes P_{<0} G)\|_{L_x^2} &\lesssim_{\mathbf{N}, \mathbf{R}, \mathbf{e}} 2^{-k} (\|\nabla F\|_{L_x^e} + \|F\|_{L_x^\infty}) \|G\|_{L_x^2}, \end{aligned}$$

where all the above norms and L-P operators are with respect to  $\gamma[v]$ .

- Given any  $v \in [0, \delta)$  and  $p \in [1, \infty]$ , we have

$$\begin{aligned} (6.8) \quad \|F \otimes G\|_{\mathcal{B}_x^0(\gamma[v])} &\lesssim_{\mathbf{N}, \mathbf{R}, \mathbf{e}} (\|\nabla F\|_{L_x^e(\gamma[v])} + \|F\|_{L_x^\infty(\gamma[v])}) \|G\|_{\mathcal{B}_x^0(\gamma[v])}, \\ \|\Phi \otimes \Psi\|_{\mathcal{B}_{t,x}^{p,0}} &\lesssim_{\mathbf{N}, \mathbf{R}, \mathbf{e}} (\|\nabla \Phi\|_{L_{t,x}^{\infty,e}} + \|\Phi\|_{L_{t,x}^{\infty,\infty}}) \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}}. \end{aligned}$$

*Proof.* The proofs are analogous to those of (3.10) and (3.12), except we use the geometric L-P theory. Thus, we only sketch the proof here. Throughout, we assume that all norms and L-P operators for  $F$  and  $G$  are with respect to  $\gamma[v]$ . Furthermore, we define constants  $\mathbf{e}'$  and  $\mathbf{e}''$  such that  $(\mathbf{e}')^{-1} = \mathbf{e}^{-1} + 2^{-1}$  and  $\mathbf{e}^{-1} + (\mathbf{e}'')^{-1} = 2^{-1}$ .

For (6.7), if  $l \geq k$ , then we “create” an instance of  $\Delta$  and apply (6.4) and (6.6):

$$\begin{aligned} \|P_k(F \otimes P_l G)\|_{L_x^2} &\lesssim 2^{-2l} \|P_k(F \otimes \tilde{\Delta} \tilde{P}_l G)\|_{L_x^2} \\ &\lesssim 2^{-2l} [\|P_k \nabla(F \otimes \tilde{P}_l G)\|_{L_x^2} + \|P_k(\nabla F \otimes \tilde{P}_l G)\|_{L_x^2}] \\ &\lesssim_{\mathbf{N}, \mathbf{R}, \mathbf{e}} 2^{-2l+k} [\|F \otimes \tilde{P}_l G\|_{L_x^2} + \|\nabla F \otimes \tilde{P}_l G\|_{L_x^{\mathbf{e}'}}]. \end{aligned}$$

Applications of Hölder’s inequality and (6.4) result in (6.7) in this case.

Next, if  $l < k$ , we apply (6.5) and Hölder’s inequality to obtain

$$\|P_k(F \otimes P_l G)\|_{L_x^2} \lesssim 2^{-k} (\|\nabla F \otimes P_l G\|_{L_x^2} + \|F \otimes \nabla P_l G\|_{L_x^2})$$

<sup>46</sup>Note also that these norms are analogous to the Besov norms used in [11, 13, 17, 18, 22, 23], which are based on the geometric L-P theory of [12]. In terms of our null cone problems of interest, the relevant Besov norms with respect to both theories have essentially the same estimates.



$$\lesssim 2^{-k}(\|\nabla F\|_{L_x^e}\|P_l G\|_{L_x^{e''}} + \|F\|_{L_x^\infty}\|\nabla P_l G\|_{L_x^2}).$$

Applying (6.4) and (6.6) results in (6.7) in this case.

Furthermore, the low-frequency estimate in (6.7) involving  $P_{<0}$  can be similarly proved using the low-frequency versions of (6.4) and (6.6).

Now, for the first estimate in (6.8), we decompose

$$\|F \otimes G\|_{\mathbb{B}_x^0} \lesssim \sum_{k,l \geq 0} \|P_k(F \otimes P_l G)\|_{L_x^2} + \sum_{k \geq 0} \|P_k(F \otimes P_{<0} G)\|_{L_x^2} + \|P_{<0}(F \otimes G)\|_{L_x^2}.$$

The last term on the right-hand side is bounded trivially:

$$\|P_{<0}(F \otimes G)\|_{L_x^2} \lesssim \|F\|_{L_x^\infty} \|G\|_{L_x^2}.$$

The remaining terms are controlled using (6.7), like in the proof of (3.12). Finally, the second estimate in (6.8) can also be proved analogously.  $\square$

**Remark.** The principal difference between (5.12) and (6.8) is that one replaces an  $L_x^2$ -norm in (5.12) by an  $L_x^e$ -norm. This weakening of the result is a consequence of not having uniform Sobolev inequalities for the  $L_x^\infty$ -norm on the  $(\mathcal{S}, \gamma[v])$ 's. To have such bounds, one needs additional curvature assumptions. <sup>47</sup>

**Remark.** Like for the main theorems in Section 5, it is easy to see that Lemma 6.3 still holds if zero or more contractions, metric contractions, and volume form contractions are added to the tensor products on the left-hand side of (6.8).

We now apply Lemma 6.3 to prove our scalar decomposition result.

**Proposition 6.4.** Assume (4.2), (4.4), and (I) hold.

- If  $v \in [0, \delta)$  and  $F \in \mathcal{C}^\infty T_l^r \mathcal{S}$ , then <sup>48</sup>

$$(6.9) \quad \|F\|_{\mathbb{B}_x^0(\gamma[v])} \simeq_{\mathbf{N}, \mathbf{R}, \mathbf{e}, C_l^r(\mathbf{d})}^c \sum_{i=1}^N \sum_{X \in \mathcal{X}_l^r(i)} \|\eta_i F(X[v])\|_{\mathbb{B}_x^0(\gamma[v])}.$$

- If  $p \in [1, \infty]$  and  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , then

$$(6.10) \quad \|\Psi\|_{\mathbb{B}_{t,x}^{p,0}} \simeq_{\mathbf{N}, \mathbf{R}, \mathbf{e}, C_l^r(\mathbf{d})}^c \sum_{i=1}^N \sum_{X \in \mathcal{X}_l^r(i)} \|\eta_i \Psi(X)\|_{\mathbb{B}_{t,x}^{p,0}}.$$

*Proof.* Here, we prove only (6.10), as the proof of (6.9) is established in an entirely analogous manner. First of all, we apply (6.8), along with the second remark after Lemma 6.3, directly for each  $\eta_i$  and  $X$  to obtain

$$\begin{aligned} \|\eta_i \Psi(X)\|_{\mathbb{B}_{t,x}^{p,0}} &\lesssim_{\mathbf{N}, \mathbf{R}, \mathbf{e}}^c (\|\nabla(\eta_i X)\|_{L_{t,x}^{\infty,e}} + \|\eta_i X\|_{L_{t,x}^{\infty,\infty}}) \|\Psi\|_{\mathbb{B}_{t,x}^{p,0}} \\ &\lesssim_{\mathbf{N}, \mathbf{R}}^c [1 + (r+l)(1+\mathbf{d})] \|\Psi\|_{\mathbb{B}_{t,x}^{p,0}}. \end{aligned}$$

In the last step, we applied (4.20) and (5.3) to control  $\eta_i X$ , as usual. Summing the above over all  $i$  and  $X$  proves one half of the estimate (6.10).

<sup>47</sup>In the null cone settings of [11, 13, 17, 18, 22, 23], one has such bounds only for scalars.

<sup>48</sup>Note that  $\eta_i X$  extends smoothly to a global horizontal field on  $\mathcal{N}$  for each  $1 \leq i \leq N$  and  $X \in \mathcal{X}_l^r(i)$ . Therefore, it makes sense to take a geometric Besov norm of  $\eta_i F(X[v])$ .

Next, given  $1 \leq i \leq N$  and  $X \in \mathcal{X}_l^r(i)$ , we let  $X^* \in \mathcal{X}_l^l(i)$  denote the dual basis element on  $U_i$  to  $X$ . Since  $\eta_i \Psi$  is supported entirely on  $U_i$ , then

$$\Psi = \sum_{i=1}^N (\eta_i \Psi) = \sum_{i=1}^N \sum_{X \in \mathcal{X}_l^r(i)} \eta_i \Psi(X) \otimes \tilde{\eta}_i X^*.$$

As a result, we can once again apply (6.8):

$$\begin{aligned} \|\Psi\|_{\mathbf{B}_{t,x}^{p,0}} &\lesssim_{\mathbf{N},\mathbf{R},\mathbf{e}}^c \sum_{i=1}^N \sum_{X \in \mathcal{X}_l^r(i)} (\|\nabla(\tilde{\eta}_i X^*)\|_{L_{t,x}^{\infty,e}} + \|\tilde{\eta}_i X^*\|_{L_{t,x}^{\infty,\infty}}) \|\eta_i \Psi(X)\|_{\mathbf{B}_{t,x}^{p,0}} \\ &\lesssim_{\mathbf{N},\mathbf{R}}^c C_l^r(d) \sum_{i=1}^N \sum_{X \in \mathcal{X}_l^r(i)} \|\eta_i \Psi(X)\|_{\mathbf{B}_{t,x}^{p,0}}. \end{aligned} \quad \square$$

**6.3. Mixed Intertwining Estimates.** We already mentioned scalar reductions in our geometric Besov norms as an important step in establishing our main Besov comparison result. The other fundamental step is a preliminary comparison for our geometric and coordinate Besov norms, in the special case of localized scalar quantities. The main technical component in this endeavor is a collection of intertwining estimates involving both geometric and coordinate L-P operators.

**Lemma 6.5.** *Assume that (4.2), (4.3), and (I) hold. Furthermore, fix  $v \in [0, \delta]$  and  $1 \leq i \leq N$ , and fix arbitrary integers  $k, l \geq 0$ .*

- If  $f \in C^\infty \mathcal{S}$ , then <sup>49</sup>

$$(6.11) \quad \begin{aligned} \|P_k[(\tilde{\eta}_i \cdot P_l f) \circ \varphi_i^{-1}]\|_{L_x^2} &\lesssim_{\mathbf{R}}^\epsilon 2^{-|k-l|} (1+b) \|P_{\sim l} f\|_{L_x^2}, \\ \|P_k[(\tilde{\eta}_i \cdot P_{<0} f) \circ \varphi_i^{-1}]\|_{L_x^2} &\lesssim_{\mathbf{R}}^\epsilon 2^{-k} \|f\|_{L_x^2}. \end{aligned}$$

- If  $g \in \mathcal{S}_x \mathbb{R}^2$ , and if  $\bar{\eta}_i = \tilde{\eta}_i \circ \varphi_i^{-1}$ , then <sup>50</sup>

$$(6.12) \quad \begin{aligned} \|P_k[(\bar{\eta}_i \cdot P_l g) \circ \varphi_i]\|_{L_x^2} &\lesssim_{\mathbf{R}}^\epsilon |k-l| 2^{-|k-l|} (1+b) \|P_{\sim l} g\|_{L_x^2}, \\ \|P_k[(\bar{\eta}_i \cdot P_{<0} g) \circ \varphi_i]\|_{L_x^2} &\lesssim_{\mathbf{R}}^\epsilon 2^{-k} \|g\|_{L_x^2}. \end{aligned}$$

In the above, all geometric norms and L-P operators are with respect to  $\gamma[v]$ .

*Proof.* Consider first the case  $l \leq k$ . For this setting, we will prove a more general version of (6.12). Let  $\nu \in \mathcal{S}_x \mathbb{R}^2$ . Applying (6.5) and (I), then

$$\begin{aligned} \|P_k(\bar{\eta}_i \nu P_l g \circ \varphi_i)\|_{L_x^2} &\lesssim 2^{-k} \|\nabla(\bar{\eta}_i \nu P_l g \circ \varphi_i)\|_{L_x^2} \\ &\lesssim^c 2^{-k} (\|\bar{\eta}_i \nu\|_{L_x^\infty} \|\partial P_l g\|_{L_x^2} + \|\partial(\bar{\eta}_i \nu)\|_{L_x^2} \|P_l g\|_{L_x^\infty}). \end{aligned}$$

Applying (3.3), (3.4), (4.6), and (4.20), we obtain, as desired,

$$(6.13) \quad \|P_k(\bar{\eta}_i \nu P_l g \circ \varphi_i)\|_{L_x^2} \lesssim_{\mathbf{R}}^\epsilon 2^{-|k-l|} (\|\partial \nu\|_{L_x^2} + \|\nu\|_{L_x^\infty}) \|P_{\sim l} g\|_{L_x^2}, \quad l \geq k.$$

In particular, by setting  $\nu \equiv 1$ , this establishes (6.12) in the  $l \leq k$  case. Furthermore, using similar reasoning, we can also obtain the “ $P_{<0}$ ”-version of (6.12).

Similarly, for (6.11), we apply (3.3), (4.20), and the (I) condition:

$$\|P_k(\tilde{\eta}_i P_l f \circ \varphi_i^{-1})\|_{L_x^2} \lesssim^c 2^{-k} (\|\nabla \tilde{\eta}_i\|_{L_x^\infty} \|P_l f\|_{L_x^2} + \|\nabla P_l f\|_{L_x^2})$$

<sup>49</sup>In (6.11), multiplying  $P_l f$  and  $P_{<0} f$  by  $\tilde{\eta}_i$  ensures that the resulting functions are compactly supported in  $U_i$ , so that the outer (classical) L-P operator  $P_k$  is well-defined.

<sup>50</sup>In (6.12), multiplying  $P_l g$  and  $P_{<0} g$  by  $\bar{\eta}_i$  ensures that the resulting functions are compactly supported in  $\varphi_i(U_i)$ , hence their compositions by  $\varphi_i$  can be smoothly extended to  $\mathcal{S}$ .

$$\lesssim_{\mathbf{R}}^c 2^{-|k-l|} \|\mathbf{P}_{\sim l} f\|_{L_x^2}.$$

Again, the “ $\mathbf{P}_{<0}$ ”-version of (6.11) can be obtained using a similar process.

Next, consider when  $l \geq k$ . Fix  $u \in \mathcal{S}_x \mathbb{R}^2$ , and  $w \in C^\infty \mathcal{S}$ , with

$$\|u\|_{L_x^2} = 1, \quad \|w\|_{L_x^2} = \|w\|_{L_x^2(\gamma[v])} = 1.$$

By standard duality arguments, it suffices to show that

$$(6.14) \quad \left| \int_{\mathbb{R}^2} P_k[(\tilde{\eta}_i \mathbf{P}_l \mathbf{P}_{\sim l} f) \circ \varphi_i^{-1}] \cdot u \right| \lesssim_{\mathbf{N}, \mathbf{R}}^c 2^{-|k-l|} (1 + \mathbf{b}) \|\mathbf{P}_{\sim l} f\|_{L_x^2},$$

$$(6.15) \quad \left| \int_{\mathcal{S}} \mathbf{P}_k[(\tilde{\eta}_i P_l P_{\sim l} g) \circ \varphi_i] \cdot w \cdot d\epsilon[v] \right| \lesssim_{\mathbf{N}, \mathbf{R}}^c |k-l| 2^{-|k-l|} (1 + \mathbf{b}) \|P_{\sim l} g\|_{L_x^2}.$$

To show this, we will need the following estimates for  $\tilde{\eta}_i \vartheta_i \mathcal{J}$ , which follows immediately from (4.5), (4.6), (4.9), (4.20), (4.21), and the **(I)** condition: <sup>51</sup>

$$(6.16) \quad \begin{aligned} \|\tilde{\eta}_i \vartheta_i \mathcal{J}\|_{L_{t,x}^{\infty}} &\lesssim^c 1, & \|\tilde{\eta}_i \vartheta_i^{-1} \mathcal{J}^{-1}\|_{L_{t,x}^{\infty}} &\lesssim^c 1, \\ \|\nabla(\tilde{\eta}_i \vartheta_i \mathcal{J})\|_{L_{x,t}^{2,\infty}} &\lesssim_{\mathbf{R}}^c \mathbf{b}, & \|\nabla(\tilde{\eta}_i \vartheta_i^{-1} \mathcal{J}^{-1})\|_{L_{x,t}^{2,\infty}} &\lesssim_{\mathbf{R}}^c \mathbf{b}. \end{aligned}$$

Let  $I_1$  and  $I_2$  denote the left-hand sides of (6.14) and (6.15), respectively. Due to the self-adjointness properties of  $\mathbf{P}_k$  (with respect to  $\gamma[v]$ ) and  $P_l$  (with respect to the standard Euclidean metric on  $\mathbb{R}^2$ ), we have

$$(6.17) \quad \begin{aligned} I_2 &= \left| \int_{\mathcal{S}} [(\tilde{\eta}_i P_l P_{\sim l} g) \circ \varphi_i] \cdot \mathcal{J}[v] \cdot \mathbf{P}_k w \cdot d\epsilon[0] \right| \\ &= \left| \int_{\mathbb{R}^2} P_l P_{\sim l} g \cdot [(\tilde{\eta}_i \mathbf{P}_k w \cdot \vartheta_i \mathcal{J}[v]) \circ \varphi_i^{-1}] \right| \\ &\leq \|P_{\sim l} g\|_{L_x^2} \|P_l[(\tilde{\eta}_i \vartheta_i \mathcal{J}[v] \cdot \mathbf{P}_k w) \circ \varphi_i^{-1}]\|_{L_x^2} \\ &= \|P_{\sim l} g\|_{L_x^2} I_{11}. \end{aligned}$$

By a similar computation, we have

$$(6.18) \quad \begin{aligned} I_1 &= \left| \int_{\mathbb{R}^2} \mathbf{P}_l \mathbf{P}_{\sim l} f \cdot \vartheta_i^{-1} \mathcal{J}^{-1}[v] \cdot \tilde{\eta}_i (P_k u \circ \varphi_i) \cdot d\epsilon[v] \right| \\ &\leq \|\mathbf{P}_{\sim l} f\|_{L_x^2} \|\mathbf{P}_l \{\vartheta_i^{-1} \mathcal{J}^{-1}[v] \cdot \tilde{\eta}_i (P_k u \circ \varphi_i)\}\|_{L_x^2}. \end{aligned}$$

Applying (6.13) in conjunction with (6.16), we obtain

$$\|\mathbf{P}_l \{\vartheta_i^{-1} \mathcal{J}^{-1}[v] \cdot \tilde{\eta}_i (P_k u \circ \varphi_i)\}\|_{L_x^2} \lesssim_{\mathbf{R}}^c 2^{-|k-l|} (1 + \mathbf{b}) \|P_{\sim k} u\|_{L_x^2} \lesssim 2^{-|k-l|} (1 + \mathbf{b}).$$

Combining the above with (6.18) yields (6.14), which completes the proof of (6.11).

Finally, to deal with (6.17), we decompose again using classical L-P operators:

$$\begin{aligned} I_{11} &\lesssim \sum_{m \geq 0} \|P_l \{\vartheta_i(\mathcal{J}[v] \circ \varphi_i^{-1}) P_m[(\tilde{\eta}_i \mathbf{P}_k w) \circ \varphi_i^{-1}]\}\|_{L_x^2} \\ &\quad + \|P_l \{\vartheta_i(\mathcal{J}[v] \circ \varphi_i^{-1}) P_{<0}[(\tilde{\eta}_i \mathbf{P}_k w) \circ \varphi_i^{-1}]\}\|_{L_x^2}. \end{aligned}$$

Applying (3.10) and (6.11), along with (6.16), then

$$I_{11} \lesssim (1 + \mathbf{b}) \left\{ \sum_{m \geq 0} 2^{-|l-m|} \|P_{\sim m}[(\tilde{\eta}_i \mathbf{P}_k w) \circ \varphi_i^{-1}]\|_{L_x^2} + 2^{-l} \|(\tilde{\eta}_i \mathbf{P}_k w) \circ \varphi_i^{-1}\|_{L_x^2} \right\}$$

<sup>51</sup>Recall (4.19) for the definition of  $\vartheta_i$ .

$$\begin{aligned} &\lesssim_{\mathbb{R}}^c (1+b) \sum_{m \geq 0} 2^{-|l-m|-|m-k|} \|\mathbf{P}_{\sim k} w\|_{L_x^2} + (1+b) 2^{-l} \|\mathbf{P}_{\sim k} w\|_{L_x^2} \\ &\lesssim (1+b) |k-l| 2^{-|l-k|}. \end{aligned}$$

The above, combined with (6.17), yields (6.15), which proves (6.12).  $\square$

**6.4. The Besov Comparison Theorem.** With the technical necessities out of the way, we can now prove our desired Besov comparison theorem. The first step is to do this in the localized scalar case. This argument is a minor variation of a similar comparison result presented in [22]. The main component of this proof is the mixed intertwining estimates proved in Lemma 6.5.

**Proposition 6.6.** *Fix  $1 \leq i \leq \mathbf{N}$ , and assume (4.2), (4.3), and (I).*

- *If  $v \in [0, \delta)$ , and if  $f \in C^\infty \mathcal{S}$  is supported within the support of  $\eta_i$ , then*

$$(6.19) \quad \|f\|_{\mathbb{B}_x^0(\gamma[v])} \simeq_{\mathbb{R},b}^c \|f \circ \varphi_i^{-1}\|_{B_x^0}.$$

- *If  $p \in [1, \infty]$ , and if  $\phi \in C^\infty \mathcal{N}$  is supported within the support of  $\epsilon \eta_i$ , then*

$$(6.20) \quad \|\phi\|_{\mathbb{B}_{t,x}^{p,0}} \simeq_{\mathbb{R},b}^c \|\phi \circ \varphi_i^{-1}\|_{B_{t,x}^{p,0}}.$$

*Proof.* Here, we will only prove (6.19), since (6.20) can be proved analogously with modifications of the intertwining estimates (6.11) and (6.12).<sup>52</sup> Here, we shall assume all relevant norms and L-P operators are defined with respect to  $\gamma[v]$ . Moreover, for notational convenience, we define  $\bar{f} = f \circ \varphi_i^{-1}$  and  $\bar{\eta} = \tilde{\eta} \circ \varphi_i^{-1}$ .

By definition, using (3.2) and (6.2), we can decompose

$$(6.21) \quad \begin{aligned} \|f\|_{\mathbb{B}_x^0} &\lesssim^c \sum_{k \geq 0} \|\mathbf{P}_k(\bar{\eta}_i \bar{f} \circ \varphi_i)\|_{L_x^2} + \|\bar{f}\|_{L_x^2} \\ &\lesssim \sum_{k,l \geq 0} \|\mathbf{P}_k(\bar{\eta}_i P_l \bar{f} \circ \varphi_i)\|_{L_x^2} + \sum_{k \geq 0} \|\mathbf{P}_k(\bar{\eta}_i P_{<0} \bar{f} \circ \varphi_i)\|_{L_x^2} + \|\bar{f}\|_{L_x^2}, \end{aligned}$$

$$(6.22) \quad \begin{aligned} \|\bar{f}\|_{B_x^0} &\lesssim^c \sum_{k \geq 0} \|P_k(\tilde{\eta}_i f \circ \varphi_i^{-1})\|_{L_x^2} + \|f\|_{L_x^2} \\ &\lesssim \sum_{k,l \geq 0} \|P_k(\tilde{\eta}_i P_l f \circ \varphi_i^{-1})\|_{L_x^2} + \sum_{k \geq 0} \|P_k(\tilde{\eta}_i P_{<0} f \circ \varphi_i^{-1})\|_{L_x^2} + \|f\|_{L_x^2}. \end{aligned}$$

Note in particular that due to the supports of  $f$  and  $\bar{f}$ , then  $\tilde{\eta}_i f = f$  and  $\bar{\eta}_i \bar{f} = \bar{f}$ .

Applying (6.11) to (6.22), then

$$\|\bar{f}\|_{B_x^0} \lesssim_{\mathbb{R},b}^c \sum_{k,l \geq 0} 2^{-|k-l|} \|\mathbf{P}_{\sim l} f\|_{L_x^2} + \sum_{k \geq 0} 2^{-k} \|f\|_{L_x^2} + \|f\|_{L_x^2} \lesssim \|f\|_{\mathbb{B}_x^0}.$$

Similarly, applying (6.12) to (6.21), then

$$\|f\|_{\mathbb{B}_x^0} \lesssim_{\mathbb{R},b}^c \sum_{k,l \geq 0} |k-l| 2^{-|k-l|} \|P_l \bar{f}\|_{L_x^2} + \sum_{k \geq 0} 2^{-k} \|\bar{f}\|_{L_x^2} + \|\bar{f}\|_{L_x^2} \lesssim \|\bar{f}\|_{B_x^0}.$$

This completes the proof of (6.19).  $\square$

Finally, we combine the localized scalar estimates of Proposition 6.6 with the scalar reduction argument of Proposition 6.4 in order to prove our main Besov comparison estimates, this time for global tensorial quantities.

**Theorem 6.7.** *Assume (4.2), (4.3), (4.4), and (I).*

<sup>52</sup>In particular, we take  $L^p$ -norms with respect to  $t$  of (6.11) and (6.12).

- If  $v \in [0, \delta)$ , and if  $F \in \mathcal{C}^\infty T_l^r \mathcal{S}$ , then

$$(6.23) \quad \|F\|_{\mathcal{B}_x^0(\gamma[v])} \simeq_{\mathbf{N}, \mathbf{R}, \mathbf{e}, \mathbf{b}, C_l^r(d)}^c \|F\|_{\mathcal{B}_x^0(\gamma[v])}.$$

- If  $p \in [1, \infty]$ , and if  $\Psi \in \mathcal{C}^\infty \underline{T}_l^r \mathcal{N}$ , then

$$(6.24) \quad \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}} \simeq_{\mathbf{N}, \mathbf{R}, \mathbf{e}, \mathbf{b}, C_l^r(d)}^c \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}}.$$

*Proof.* To show (6.24), we simply apply the scalar reduction estimate (6.10) followed by the localized scalar comparison (6.20):

$$\begin{aligned} \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}} &\simeq_{\mathbf{N}, \mathbf{R}, \mathbf{e}, C_l^r(d)}^c \sum_{i=1}^N \sum_{X \in \mathcal{X}_l^r(i)} \|\eta_i \Psi(X)\|_{\mathcal{B}_{t,x}^{p,0}} \\ &\simeq_{\mathbf{R}, \mathbf{b}}^c \sum_{i=1}^N \sum_{X \in \mathcal{X}_l^r(i)} \|\eta_i \Psi(X) \circ \varphi_i^{-1}\|_{\mathcal{B}_{t,x}^{p,0}} \\ &= \|\Psi\|_{\mathcal{B}_{t,x}^{p,0}}. \end{aligned}$$

The proof of (6.23) is entirely analogous.  $\square$

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